

# A Master Action for $D = 11$ Supergravity in the Component Formulation

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## Abstract

We give a solution to the classical master equation of  $D = 11$  supergravity in the conventional component formulation. Based on a careful investigation of the symmetry algebra including terms proportional to the equation of motion, we construct an explicit expression of the master action in an order-by-order manner.

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# 1 Introduction

Supersymmetric field theories often show tractableness of quantum behavior. One of them is the cancellation of loop divergences. Especially this property has been studied in the most supersymmetric theory i.e.  $D = 11$  supergravity and its dimensional reductions. A partial list of literature on recent discussions on this subject is [1]-[6].

Since perturbative calculation of supergravity amplitudes is quite complicated, the on-shell condition for external states is often imposed to simplify the calculation and to use various methods for circumventing the complexity of direct calculation. If we want to perform covariant off-shell calculation of loop amplitudes in the standard way based on Feynman rules, we need to introduce ghosts and fix the local symmetry. Because the local symmetry of  $D = 11$  supergravity is reducible and its algebra is open, Faddeev-Popov gauge fixing procedure does not work, and we have to use field-antifield formalism, or Batalin-Vilkovisky (BV) formalism (For reviews, see e.g. [7]-[11]). In this formalism additional fields are introduced and using them it is necessary to construct a master action which satisfies the classical master equation. For  $D = 11$  supergravity, this has been done in the pure spinor superfield formulation in [12]. In this formulation the master action takes very simple form. However, as a cost of the simplification, superfields contain huge number of auxiliary fields, and the relation between them and the conventional component expression is not immediately clear.

In this paper we give a master action for  $D = 11$  supergravity in conventional component expression. Although introduction of ghosts makes the Feynman rules even more complicated and it may not be practical to use the action for computing amplitudes, it expresses the relation between structure functions of the symmetry algebra compactly, and it may be useful for formal arguments about properties of amplitudes. After quickly reviewing  $D = 11$  supergravity and fixing the notation in Section 2 and Appendix A, we introduce ghosts and investigate Jacobi identity in Section 3. Then we construct a master action in Section 4. Section 5 contains a conclusion. As is usual in supergravity theories, we need tedious calculation, especially for Fierz transformation. Such calculations are made with the help of symbolic manipulation program Mathematica and the package for gamma-matrix algebra GAMMA[13]. We give the outlines of the calculation in Appendix B, C, and D.

After this work was finished, we realized that [14] has already studied component expression of the field-antifield formulation of  $D = 11$  supergravity.

## 2 $D = 11$ supergravity and its local symmetries

The action of  $D = 11$  supergravity  $S_0 = \frac{1}{2\kappa^2}\mathcal{S}_0$ , which consists of the vielbein  $e_\mu^a$ , the gravitino  $\psi_\mu^\alpha$ , and the 3-form  $A_{\mu\nu\lambda}$ , is given by

$$\begin{aligned}\mathcal{S}_0 = & \int d^{11}x e \left[ R(\omega) - \frac{i}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\lambda} D_\nu \left( \frac{1}{2} (\omega + \hat{\omega}) \right) \psi_\lambda - \frac{1}{2 \cdot 4!} F_{\mu_1 \dots \mu_4} F^{\mu_1 \dots \mu_4} \right. \\ & - \frac{i}{192} (\bar{\psi}_{\nu_1} \Gamma^{\nu_1 \mu_1 \dots \mu_4 \nu_2} \psi_{\nu_2} + 12 \bar{\psi}^{\mu_1} \Gamma^{\mu_2 \mu_3} \psi^{\mu_4}) \cdot \frac{1}{2} (\hat{F}_{\mu_1 \dots \mu_4} + F_{\mu_1 \dots \mu_4}) \\ & \left. + \frac{\sigma}{(144)^2} \epsilon^{\mu_1 \dots \mu_{11}} A_{\mu_1 \mu_2 \mu_3} F_{\mu_4 \dots \mu_7} F_{\mu_8 \dots \mu_{11}} \right],\end{aligned}\quad (2.1)$$

where

$$\begin{aligned}\omega_{\mu ab} &= \omega_{\mu ab}(e) + \frac{1}{2} (T_{\mu ab} - T_{ab\mu} - T_{b\mu a}), \\ e_\nu^a e_\lambda^b \omega_{\mu ab}(e) &= e_{\nu a} \partial_{[\lambda} e_{\mu]}^a + e_{\lambda a} \partial_{[\mu} e_{\nu]}^a - e_{\mu a} \partial_{[\nu} e_{\lambda]}^a, \\ T^\mu{}_{ab} &= \frac{i}{4} \bar{\psi}_{[a} \Gamma^\mu \psi_{b]} - \frac{i}{8} \bar{\psi}_\nu \Gamma_{ab}{}^{\nu\lambda} \psi_\lambda, \\ F_{\mu_1 \dots \mu_4} &= 4 \partial_{[\mu_1} A_{\mu_2 \mu_3 \mu_4]}, \\ \hat{\omega}_{\mu ab} &= \omega_{\mu ab} - \frac{i}{16} \bar{\psi}_\nu \Gamma_{ab\mu}{}^{\nu\lambda} \psi_\lambda, \\ \hat{F}_{\mu_1 \dots \mu_4} &= F_{\mu_1 \dots \mu_4} + \frac{3}{2} i \bar{\psi}_{[\mu_1} \Gamma_{\mu_2 \mu_3} \psi_{\mu_4]},\end{aligned}\quad (2.2)$$

and  $\sigma = \pm 1$  is the sign factor defining the 10th gamma matrix (see (A.2)).  $T^\mu{}_{ab}$  is the torsion determined by 1.5 order formalism. For our notation about spinors see Appendix A.

This action has four local symmetries: the supersymmetry, the diffeomorphism, the local Lorentz symmetry, and the 3-form gauge symmetry. The supersymmetry transformation  $\delta^S$  is given by

$$\delta_\xi^S e_\mu^a = \frac{i}{4} \bar{\xi} \Gamma^a \psi_\mu, \quad \delta_\xi^S e_a{}^\mu = -\frac{i}{4} \bar{\xi} \Gamma^\mu \psi_a, \quad (2.3)$$

$$\delta_\xi^S \psi_\mu = \tilde{D}_\mu \xi, \quad \delta_\xi^S \bar{\psi}_\mu = \tilde{D}_\mu \bar{\xi}, \quad (2.4)$$

$$\delta_\xi^S A_{\mu\nu\lambda} = -\frac{3}{4} i \bar{\xi} \Gamma_{[\mu\nu} \psi_{\lambda]}, \quad (2.5)$$

where the parameter  $\xi$  is a Majorana spinor, and

$$\begin{aligned}\tilde{D}_\mu \xi &:= D_\mu(\hat{\omega}) \xi + \frac{1}{288} (\Gamma^{\mu_1 \dots \mu_4}{}_\mu - 8 \delta_\mu{}^{\mu_1} \Gamma^{\mu_2 \mu_3 \mu_4}) \xi \hat{F}_{\mu_1 \dots \mu_4}, \\ \tilde{D}_\mu \bar{\xi} &:= D_\mu(\hat{\omega}) \bar{\xi} - \frac{1}{288} \bar{\xi} (\Gamma^{\mu_1 \dots \mu_4}{}_\mu + 8 \delta_\mu{}^{\mu_1} \Gamma^{\mu_2 \mu_3 \mu_4}) \hat{F}_{\mu_1 \dots \mu_4}.\end{aligned}\quad (2.6)$$

The diffeomorphism  $\delta^D$ , the local Lorentz transformation  $\delta^L$ , and the 3-form gauge transformation  $\delta^A$  take the standard form:

$$\begin{aligned}\delta_\epsilon^D e_\mu^a &= -\epsilon^\nu \partial_\nu e_\mu^a - \partial_\mu \epsilon^\nu e_\nu^a, \\ \delta_\epsilon^D \psi_\mu^\alpha &= -\epsilon^\nu \partial_\nu \psi_\mu^\alpha - \partial_\mu \epsilon^\nu \psi_\nu^\alpha, \\ \delta_\epsilon^D A_{\mu\nu\lambda} &= -\epsilon^\rho \partial_\rho A_{\mu\nu\lambda} - 3\partial_{[\mu} \epsilon^\rho A_{|\rho|\nu\lambda]},\end{aligned}\tag{2.7}$$

$$\delta_\lambda^L e_\mu^a = \lambda^a{}_b e_\mu^b, \quad \delta_\lambda^L \psi_\mu = \frac{1}{4} \lambda_{ab} \Gamma^{ab} \psi_\mu, \quad \delta_\lambda^L A_{\mu\nu\lambda} = 0,\tag{2.8}$$

$$\delta_\theta^A e_\mu^a = 0, \quad \delta_\theta^A \psi_\mu = 0, \quad \delta_\theta^A A_{\mu\nu\lambda} = 3\partial_{[\mu} \theta_{\nu\lambda]}.\tag{2.9}$$

Hatted fields  $\hat{\omega}_{\mu ab}$  and  $\hat{F}_{\mu\nu\lambda\rho}$  are supercovariant i.e. their supersymmetry transformation do not contain derivatives of the parameter:

$$\begin{aligned}\delta_\xi^S \hat{\omega}_{\mu ab} &= \frac{i}{4} e_a{}^\nu e_b{}^\lambda \left[ \bar{\xi} \Gamma_{[\nu} \tilde{D}_{\lambda]} \psi_\mu + \bar{\xi} \Gamma_{[\lambda} \tilde{D}_{|\mu|} \psi_{\nu]} - \bar{\xi} \Gamma_\mu \tilde{D}_{[\nu} \psi_{\lambda]} \right. \\ &\quad \left. - \frac{1}{144} \bar{\xi} (\Gamma_{\nu\lambda}{}^{\mu_1 \dots \mu_4} + 24 \delta_{[\nu}{}^{\mu_1} \delta_{\lambda]}{}^{\mu_2} \Gamma^{\mu_3 \mu_4}) \psi_\mu \hat{F}_{\mu_1 \dots \mu_4} \right],\end{aligned}\tag{2.10}$$

$$\delta_\xi^S \hat{F}_{\mu\nu\lambda\rho} = -3i \bar{\xi} \Gamma_{[\mu\nu} \tilde{D}_{\lambda]} \psi_{\rho]} - i \bar{\xi} \Gamma^\sigma \psi_{[\mu} \hat{F}_{\nu\lambda\rho]\sigma}.\tag{2.11}$$

Note that in the above expressions all the  $O(\psi^3)$  terms are hidden in  $\tilde{D}_\mu$  and  $\hat{F}_{\mu_1 \dots \mu_4}$ . To show that the explicit  $O(\psi^3)$  terms are canceled for  $\delta_\xi^S \hat{F}_{\mu\nu\lambda\rho}$  we need Fierz identity (A.10).

Taking variation of the action with respect to  $\psi_\mu$ , we obtain

$$\delta \mathcal{S}_0 = \int d^{11}x (-ie) \delta \bar{\psi}_\mu \Gamma^{\mu\nu\lambda} \tilde{D}_\nu \psi_\lambda.\tag{2.12}$$

In this expression all the  $O(\psi^3)$  terms are hidden in  $\tilde{D}_\mu$ . To show that the explicit  $O(\psi^3)$  terms are canceled we need Fierz identity (A.13). Then the equation of motion of  $\psi_\mu$  is

$$\begin{aligned}0 &= \Gamma^{\mu\nu\lambda} \tilde{D}_\nu \psi_\lambda \\ &= \Gamma^{\mu\nu\lambda} \left[ D_\nu (\hat{\omega}) \psi_\lambda + \frac{1}{288} (\Gamma^{\mu_1 \dots \mu_4}{}_\nu - 8 \delta_\nu{}^{\mu_1} \Gamma^{\mu_2 \mu_3 \mu_4}) \psi_\lambda \hat{F}_{\mu_1 \dots \mu_4} \right].\end{aligned}\tag{2.13}$$

Commutators of the local symmetries except the one between two supersymmetries are given as follows:

$$[\delta_{\theta_1}^A, \delta_{\theta_2}^A] = 0, \quad [\delta_\theta^A, \delta_\lambda^L] = 0, \quad [\delta_\theta^A, \delta_\xi^S] = 0,\tag{2.14}$$

$$[\delta_\theta^A, \delta_\epsilon^D] = \delta_{\theta'}^A, \quad \theta'_{\mu\nu} = -3\epsilon^\lambda \partial_{[\lambda} \theta_{\mu\nu]},\tag{2.15}$$

$$[\delta_{\lambda_1}^L, \delta_{\lambda_2}^L] = \delta_{\lambda_{12}}^L, \quad \lambda_{12}^{ab} = -[\lambda_1, \lambda_2]^{ab} = -\lambda_1^a{}_c \lambda_2^{cb} + \lambda_2^a{}_c \lambda_1^{cb},\tag{2.16}$$

$$[\delta_\lambda^L, \delta_\epsilon^D] = \delta_\lambda^L, \quad \lambda'_{ab} = -\epsilon^\mu \partial_\mu \lambda_{ab}, \quad (2.17)$$

$$[\delta_\lambda^L, \delta_\xi^S] = \delta_\xi^S, \quad \xi' = -\frac{1}{4} \lambda_{ab} \Gamma^{ab} \xi, \quad (2.18)$$

$$[\delta_{\epsilon_1}^D, \delta_{\epsilon_2}^D] = \delta_{\epsilon_{12}}^D, \quad \epsilon_{12}^\mu = [\epsilon_1, \epsilon_2]^\mu = \epsilon_1^\nu \partial_\nu \epsilon_2^\mu - \epsilon_2^\nu \partial_\nu \epsilon_1^\mu, \quad (2.19)$$

$$[\delta_\epsilon^D, \delta_\xi^S] = \delta_\xi^S, \quad \xi'' = \epsilon^\mu \partial_\mu \xi. \quad (2.20)$$

We see that the above commutators are closed i.e. they are expressed by linear combinations of the four local symmetry transformations. However the commutator between two supersymmetries is not closed:

$$[\delta_{\xi_1}^S, \delta_{\xi_2}^S] = \delta_\epsilon^D + \delta_\lambda^L + \delta_\xi^S + \delta_\theta^A + \delta^t, \quad (2.21)$$

where

$$\epsilon^\mu = \frac{i}{4} \bar{\xi}_1 \Gamma^\mu \xi_2, \quad (2.22)$$

$$\lambda_{ab} = -\epsilon^\mu \hat{\omega}_{\mu ab} - \frac{i}{576} \bar{\xi}_1 (\Gamma_{ab}^{\mu_1 \dots \mu_4} \hat{F}_{\mu_1 \dots \mu_4} + 24 \Gamma^{\mu\nu} \hat{F}_{ab\mu\nu}) \xi_2, \quad (2.23)$$

$$\xi = \epsilon^\mu \psi_\mu, \quad (2.24)$$

$$\theta_{\mu\nu} = \epsilon^\lambda A_{\lambda\mu\nu} + \frac{i}{4} \bar{\xi}_1 \Gamma_{\mu\nu} \xi_2, \quad (2.25)$$

and the ‘trivial symmetry’  $\delta^t$ , which is proportional to the equation of motion of  $\psi_\mu$ , is given by

$$\delta^t e_\mu{}^a = 0, \quad \delta^t A_{\mu\nu\lambda} = 0, \quad (2.26)$$

and

$$\begin{aligned} \delta^t \psi_\mu &= \frac{i}{16} \bar{\xi}_1 \Gamma^\nu \xi_2 \left[ -\frac{5}{12} g_{\nu\mu} \Gamma_\lambda - \frac{5}{12} g_{\nu\lambda} \Gamma_\mu + \frac{3}{2} g_{\mu\lambda} \Gamma_\nu + \frac{29}{144} \Gamma_\mu \Gamma_\nu \Gamma_\lambda \right] \Gamma^{\lambda\lambda_1\lambda_2} \tilde{D}_{\lambda_1} \psi_{\lambda_2} \\ &+ \frac{i}{32} \bar{\xi}_1 \Gamma^{\nu_1\nu_2} \xi_2 \left[ \frac{7}{2} g_{\mu\nu_1} g_{\lambda\nu_2} - \frac{1}{4} g_{\mu\lambda} \Gamma_{\nu_1\nu_2} \right. \\ &+ \frac{1}{3} g_{\lambda\nu_1} \Gamma_\mu \Gamma_{\nu_2} - \frac{1}{3} g_{\mu\nu_1} \Gamma_{\nu_2} \Gamma_\lambda + \frac{7}{144} \Gamma_\mu \Gamma_{\nu_1\nu_2} \Gamma_\lambda \left. \right] \Gamma^{\lambda\lambda_1\lambda_2} \tilde{D}_{\lambda_1} \psi_{\lambda_2} \\ &+ \frac{i}{384} \bar{\xi}_1 \Gamma^{\nu_1 \dots \nu_5} \xi_2 \left[ -g_{\mu\nu_1} g_{\lambda\nu_2} \Gamma_{\nu_3\nu_4\nu_5} - \frac{1}{12} g_{\mu\nu_1} \Gamma_{\nu_2 \dots \nu_5} \Gamma_\lambda - \frac{1}{12} g_{\lambda\nu_1} \Gamma_\mu \Gamma_{\nu_2 \dots \nu_5} \right. \\ &+ \left. \frac{1}{144} \Gamma_\mu \Gamma_{\nu_1 \dots \nu_5} \Gamma_\lambda \right] \Gamma^{\lambda\lambda_1\lambda_2} \tilde{D}_{\lambda_1} \psi_{\lambda_2}. \end{aligned} \quad (2.27)$$

To show (2.21) on  $A_{\mu\nu\lambda}$  we need Fierz identity (A.10).

Thus the symmetry algebra is open. Moreover the 3-form gauge transformation is reducible i.e. the transformation parameter also has a ‘local symmetry’  $\delta\theta_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]}$ . Again  $\theta_\mu$  also has a ‘local symmetry’  $\delta\theta_\mu = \partial_\mu\theta$ . Therefore to perform gauge fixing of those local symmetries we must use field-antifield formalism.

### 3 Ghosts and commutators

In this section we introduce ghosts into  $D = 11$  supergravity following the field-antifield formalism (mainly following the description in [11]), and investigate Jacobi identity of the local symmetry algebra.

$e_\mu^a, \psi_\mu^\alpha$ , and  $A_{\mu\nu\lambda}$  in the original supergravity action  $S_0$  are denoted collectively by  $C^{A-1}$ , where indices  $A_{-1}, B_{-1}, \dots$  denote three types of fields  $(e), (\psi)$ , and  $(A)$ :

$$\begin{aligned} (C^{A-1}) &= (C^{(e)}, C^{(\psi)}, C^{(A)}) \\ &= (C^{(\mu a)}, C^{(\mu \alpha)}, C^{[\mu\nu\lambda]}) = (e_\mu^a, \psi_\mu^\alpha, A_{\mu\nu\lambda}). \end{aligned} \quad (3.1)$$

In our notation, indices implicitly contain spacetime positions, and contractions of such indices contain integrations of the positions. Usually we do not have to be conscious of the presence of these integrations and we can think of indices as those taking discrete values. However when derivative operators are involved we have to deal with them carefully, as is done in Appendix B. In this notation, the equation of motion of  $\psi_\mu$  is

$$0 = \partial \mathcal{S}_0 / \partial \psi_\mu^\alpha = -ie(C^{-1} \Gamma^{\mu\nu\lambda} \tilde{D}_\nu \psi_\lambda)_\alpha. \quad (3.2)$$

The infinitesimal local symmetry transformation with parameter  $\epsilon^{A_0}$  is denoted by

$$\delta_\epsilon C^{A-1} = R^{A-1}_{A_0} [C^{B-1}] \epsilon^{A_0}. \quad (3.3)$$

$R^{A-1}_{A_0}$  may contain derivative operators, and depend on  $C^{B-1}$ . Explicit expressions of  $R^{A-1}_{A_0}$  are readily read off from (2.3)-(2.9). Indices  $A_0, B_0, \dots$  denote four types of symmetries  $(A), (L), (D)$  and  $(S)$ :

$$\begin{aligned} (\epsilon^{A_0}) &= (\epsilon^{(A)}, \epsilon^{(L)}, \epsilon^{(D)}, \epsilon^{(S)}) \\ &= (\epsilon^{[\mu\nu]}, \epsilon^{[ab]}, \epsilon^{(\mu)}, \epsilon^{(\alpha)}) = (\theta_{\mu\nu}, \lambda^{ab}, \epsilon^\mu, \xi^\alpha). \end{aligned} \quad (3.4)$$

The action  $\mathcal{S}_0$  is invariant under the transformation:

$$\partial \mathcal{S}_0 / \partial C^{A-1} R^{A-1}_{A_0} = 0. \quad (3.5)$$

Corresponding to the symmetry, we introduce ghosts  $C^{A_0}$ :

$$\begin{aligned} (C^{A_0}) &= (C^{[\mu\nu]}, C^{[ab]}, C^{(\mu)}, C^{(\alpha)}) \\ &= (c_{\mu\nu}, c^{ab}, c^\mu, c^\alpha). \end{aligned} \quad (3.6)$$

To avoid confusion,  $e_{\mu a} e_{\nu b} c^{ab}$  will never be denoted by  $c_{\mu\nu}$ . Since the local symmetry is reducible i.e. the symmetry parameter  $\theta_{\mu\nu}$  has a ‘symmetry’, there exist such  $R^{A_0}_{A_1}$  and  $R^{A_1}_{A_2}$  that

$$R^{A_{-1}}_{A_0} R^{A_0}_{A_1} = 0, \quad R^{A_0}_{A_1} R^{A_1}_{A_2} = 0. \quad (3.7)$$

Correspondingly we introduce a ‘ghost of ghost’  $C^{A_1}$  and a ‘ghost of ghost of ghost’  $C^{A_2}$ . Indices  $A_1$  and  $A_2$  take only one type of fields respectively:  $A_1 = [\mu]$ , and  $A_2$  is empty:

$$(C^{A_1}) = (C^{[\mu]}) = (C_\mu), \quad (C^{A_2}) = (C). \quad (3.8)$$

$R^{A_0}_{A_1}$  is nonzero only when  $A_0 = (A) = [\mu\nu]$ . For  $A_0 = [\mu\nu]$  and  $A_1 = [\mu]$ , explicit expressions of  $R^{A_0}_{A_1}$  and  $R^{A_1}_{A_2}$  are given by

$$R^{A_0}_{B_1} C^{B_1} = 2\partial_{[\mu} C_{\nu]}, \quad R^{A_1}_{B_2} C^{B_2} = \partial_\mu C, \quad (3.9)$$

and note that  $R^{A_0}_{A_1}$  and  $R^{A_1}_{A_2}$  do not depend on fields.

We assign statistical parity  $st$  and ghost number  $gh$  to each field as follows:

$$st[C^{A_n}] = A_n + n + 1, \quad gh[C^{A_n}] = n + 1. \quad (3.10)$$

As usual,  $st[f] = 0(= 1) \bmod 2$  means that  $f$  is commuting (anticommuting). Statistical parity of an index  $A_n$  is denoted by  $A_n$  itself. If  $A_n$  contains a spinor index, then  $A_n = 1$ , otherwise  $A_n = 0$ . We will often use this notation in sign factors, especially in powers of  $(-1)$ .

The commutator of two local symmetries is

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}] C^{A_{-1}} &= \left[ \partial R^{A_{-1}}_{B_0} / \partial C^{B_{-1}} R^{B_{-1}}_{C_0} \right. \\ &\quad \left. - (-1)^{B_0 C_0} \partial R^{A_{-1}}_{C_0} / \partial C^{B_{-1}} R^{B_{-1}}_{B_0} \right] \epsilon_1^{C_0} \epsilon_2^{B_0}, \end{aligned} \quad (3.11)$$

and the right hand side must be expressed by linear combination of the local symmetries and ‘trivial symmetry’ proportional to the equation of motion  $\partial \mathcal{S}_0 / \partial C^{A_{-1}}$ :

$$\begin{aligned} \partial R^{A_{-1}}_{B_0} / \partial C^{B_{-1}} R^{B_{-1}}_{C_0} - (-1)^{B_0 C_0} \partial R^{A_{-1}}_{C_0} / \partial C^{B_{-1}} R^{B_{-1}}_{B_0} \\ = R^{A_{-1}}_{A_0} T^{A_0}_{B_0 C_0} + \partial \mathcal{S}_0 / \partial C^{B_{-1}} E^{B_{-1} A_{-1}}_{B_0 C_0}, \end{aligned} \quad (3.12)$$

where  $T^{A_0}_{B_0 C_0}$  and  $E^{B_{-1} A_{-1}}_{B_0 C_0}$  has graded antisymmetry in  $(B_0, C_0)$  and  $(B_{-1}, A_{-1})$ :

$$\begin{aligned} T^{A_0}_{B_0 C_0} &= (-1)^{1+B_0 C_0} T^{A_0}_{C_0 B_0}, \\ E^{B_{-1} A_{-1}}_{B_0 C_0} &= (-1)^{1+A_{-1} B_{-1}} E^{A_{-1} B_{-1}}_{B_0 C_0} \end{aligned} \quad (3.13)$$

$$= (-1)^{1+B_0C_0} E^{B_{-1}A_{-1}}_{C_0B_0}. \quad (3.14)$$

$T^{A_0}_{B_0C_0}$  is the ‘structure constant’ of this symmetry, and its definition has an ambiguity: If we add  $R^{A_0}_{A_1} \tilde{T}^{A_1}_{B_0C_0}$  to  $T^{A_0}_{B_0C_0}$ , (3.12) is still satisfied. An explicit form of  $T^{A_0}_{B_0C_0}$  up to this ambiguity can be read off from (2.14)-(2.21). Defining  $T^{A_0}$  as

$$T^{A_0} := (-1)^{B_0} T^{A_0}_{B_0C_0} C^{C_0} C^{B_0}, \quad (3.15)$$

and  $\bar{c}_\alpha := -(c^T \mathcal{C}^{-1})_\alpha$ , components of  $T^{A_0}$  are

$$T^{[\mu\nu]} = 6c^\lambda \partial_{[\lambda} c_{\mu\nu]} - \frac{i}{4} \bar{c} \Gamma^\lambda c A_{\lambda\mu\nu} - \frac{i}{4} \bar{c} \Gamma_{\mu\nu} c, \quad (3.16)$$

$$\begin{aligned} T^{[ab]} &= -2c^a c^{cb} + 2c^\mu \partial_\mu c^{ab} + \frac{i}{4} \bar{c} \Gamma^\mu c \hat{\omega}_\mu^{ab} \\ &\quad + \frac{i}{576} \bar{c} (\Gamma^{ab\mu_1 \dots \mu_4} + 24e^{a\mu_1} e^{b\mu_2} \Gamma^{\mu_3\mu_4}) c \hat{F}_{\mu_1 \dots \mu_4}, \end{aligned} \quad (3.17)$$

$$T^{(\mu)} = 2c^\nu \partial_\nu c^\mu - \frac{i}{4} \bar{c} \Gamma^\mu c, \quad (3.18)$$

$$T^{(\alpha)} = \frac{1}{2} c_{ab} (\Gamma^{ab} c)^\alpha - 2c^\mu \partial_\mu c^\alpha - \frac{i}{4} \bar{c} \Gamma^\mu c \psi_\mu^\alpha, \quad (3.19)$$

where the ambiguity is fixed so that  $T^{A_0}$  contains  $c_{\mu\nu}$  only in the form of its field strength. Similarly  $E^{B_{-1}A_{-1}}_{B_0C_0}$  can be read off from (2.26) and (2.27).  $E^{B_{-1}A_{-1}}_{B_0C_0}$  also has an ambiguity: if we add  $\partial \mathcal{S}_0 / \partial C^{C_{-1}} \tilde{E}^{C_{-1}B_{-1}A_{-1}}_{B_0C_0}$  with  $(C_{-1}, B_{-1})$  gradedly antisymmetrized to  $E^{B_{-1}A_{-1}}_{B_0C_0}$ , (3.12) is still satisfied. For simplicity we fix this kind of ambiguity so that no terms proportional to derivatives of fields appear. Then defining  $E^{B_{-1}A_{-1}}$  as

$$E^{B_{-1}A_{-1}} := (-1)^{B_0} E^{B_{-1}A_{-1}}_{B_0C_0} C^{C_0} C^{B_0}, \quad (3.20)$$

explicit forms of nonzero components of  $E^{B_{-1}A_{-1}}$  are given by

$$\begin{aligned} E^{(\nu\beta)(\mu\alpha)} &= \frac{1}{16} e^{-1} \left[ \bar{c} \Gamma^a c \left\{ -\frac{5}{12} e_{\mu a} \Gamma_\nu \mathcal{C} - \frac{5}{12} e_{\nu a} \Gamma_\mu \mathcal{C} + \frac{3}{2} e_{\mu\nu} \Gamma_a \mathcal{C} + \frac{29}{144} \Gamma_\mu \Gamma_a \Gamma_\nu \mathcal{C} \right\}^{\alpha\beta} \right. \\ &\quad + \frac{1}{2} \bar{c} \Gamma^{a_1 a_2} c \left\{ \frac{7}{2} e_{\mu a_1} e_{\nu a_2} \mathcal{C} - \frac{1}{4} g_{\mu\nu} \Gamma_{a_1 a_2} \mathcal{C} \right. \\ &\quad + \frac{1}{3} e_{\nu a_1} \Gamma_\mu \Gamma_{a_2} \mathcal{C} - \frac{1}{3} e_{\mu a_1} \Gamma_{a_2} \Gamma_\nu \mathcal{C} + \frac{7}{144} \Gamma_\mu \Gamma_{a_1 a_2} \Gamma_\nu \mathcal{C} \left. \right\}^{\alpha\beta} \\ &\quad + \frac{1}{24} \bar{c} \Gamma^{a_1 \dots a_5} c \left\{ -e_{\mu a_1} e_{\nu a_2} \Gamma_{a_3 a_4 a_5} \mathcal{C} \right. \\ &\quad \left. - \frac{1}{12} e_{\mu a_1} \Gamma_{a_2 \dots a_5} \Gamma_\nu \mathcal{C} - \frac{1}{12} e_{\nu a_1} \Gamma_\mu \Gamma_{a_2 \dots a_5} \mathcal{C} + \frac{1}{144} \Gamma_\mu \Gamma_{a_1 \dots a_5} \Gamma_\nu \mathcal{C} \right\}^{\alpha\beta} \left. \right]. \end{aligned} \quad (3.21)$$

Note that  $E^{B_{-1}A_{-1}}_{B_0C_0}$  has the following properties, which will often be used later:

$$E^{(\psi)(\psi)}_{(S)(S)} \text{ is the only nonzero component}$$



$$\text{and it depends only on } e_\mu^a. \quad (3.22)$$

Next let us investigate Jacobi identity

$$([\delta_{\epsilon_1}, [\delta_{\epsilon_2}, \delta_{\epsilon_3}]] + [\delta_{\epsilon_2}, [\delta_{\epsilon_3}, \delta_{\epsilon_1}]] + [\delta_{\epsilon_3}, [\delta_{\epsilon_1}, \delta_{\epsilon_2}]])C^{A-1} = 0. \quad (3.23)$$

The left hand side can be calculated using (3.11) and (3.12), and we obtain

$$0 = R^{A-1}_{A_0} A^{A_0}_{B_0 C_0 D_0} + \partial S_0 / \partial C^{B-1} B^{B-1 A-1}_{B_0 C_0 D_0}, \quad (3.24)$$

where

$$A^{A_0}_{B_0 C_0 D_0} = \partial T^{A_0}_{[B_0 C_0]} / \partial C^{A-1} R^{A-1}_{D_0\}} - T^{A_0}_{[B_0|E_0|} T^{E_0}_{C_0 D_0\}}, \quad (3.25)$$

$$\begin{aligned} B^{B-1 A-1}_{B_0 C_0 D_0} &= \partial E^{B-1 A-1}_{[B_0 C_0]} / \partial C^{D-1} R^{D-1}_{D_0\}} - E^{B-1 A-1}_{[B_0|E_0|} T^{E_0}_{C_0 D_0\}} \\ &\quad - (-1)^{A-1 B_0} \partial R^{B-1}_{[B_0]} / \partial C^{D-1} E^{D-1 A-1}_{C_0 D_0\}} \\ &\quad + (-1)^{A-1 B-1+B_0} \partial R^{A-1}_{[B_0]} / \partial C^{D-1} E^{D-1 B-1}_{C_0 D_0\}}, \end{aligned} \quad (3.26)$$

and  $[B_0 C_0 D_0\}$  means graded antisymmetrization. It is understood that if there is a sign factor dependent on these indices, we also interchange the indices in the sign factor. For example,

$$\begin{aligned} &(-1)^{A-1 B_0} \partial R^{B-1}_{[B_0]} / \partial C^{D-1} E^{D-1 A-1}_{C_0 D_0\}} \\ &= \frac{1}{3} \left[ (-1)^{A-1 B_0} \partial R^{B-1}_{B_0} / \partial C^{D-1} E^{D-1 A-1}_{C_0 D_0} \right. \\ &\quad + (-1)^{A-1 C_0+B_0(C_0+D_0)} \partial R^{B-1}_{C_0} / \partial C^{D-1} E^{D-1 A-1}_{D_0 B_0} \\ &\quad \left. + (-1)^{A-1 D_0+D_0(B_0+C_0)} \partial R^{B-1}_{D_0} / \partial C^{D-1} E^{D-1 A-1}_{B_0 C_0} \right]. \end{aligned} \quad (3.27)$$

To satisfy (3.24),  $A^{A_0}_{B_0 C_0 D_0}$  and  $B^{B-1 A-1}_{B_0 C_0 D_0}$  must be in the following form (see e.g. [11]):

$$A^{A_0}_{B_0 C_0 D_0} = R^{A_0}_{A_1} F^{A_1}_{B_0 C_0 D_0} + \partial S_0 / \partial C^{B-1} D^{B-1 A_0}_{B_0 C_0 D_0}, \quad (3.28)$$

$$\begin{aligned} B^{B-1 A-1}_{B_0 C_0 D_0} &= (-1)^{A-1 A_0} R^{B-1}_{A_0} D^{A-1 A_0}_{B_0 C_0 D_0} \\ &\quad + (-1)^{1+B-1(A-1+A_0)} R^{A-1}_{A_0} D^{B-1 A_0}_{B_0 C_0 D_0} \\ &\quad + \partial S_0 / \partial C^{C-1} M^{C-1 B-1 A-1}_{B_0 C_0 D_0}, \end{aligned} \quad (3.29)$$

where  $F^{A_1}_{B_0 C_0 D_0}$ ,  $D^{A-1 A_0}_{B_0 C_0 D_0}$  and  $M^{C-1 B-1 A-1}_{B_0 C_0 D_0}$  has graded antisymmetry in  $(B_0, C_0, D_0)$  and  $(C_{-1}, B_{-1}, A_{-1})$ . The definition of  $F^{A_1}_{B_0 C_0 D_0}$  has an ambiguity: If we add  $R^{A_1}_{A_2} \tilde{F}^{A_2}_{B_0 C_0 D_0}$

to  $F^{A_1}_{B_0 C_0 D_0}$ , (3.28) is still satisfied.  $D^{B_{-1} A_0}_{B_0 C_0 D_0}$  and  $M^{C_{-1} B_{-1} A_{-1}}_{B_0 C_0 D_0}$  also has an ambiguity similar to  $E^{B_{-1} A_{-1}}_{B_0 C_0}$ , which will be fixed similarly.

By computing the expression (3.25) and (3.26) explicitly, we can confirm that  $A^{A_0}_{B_0 C_0 D_0}$  and  $B^{B_{-1} A_{-1}}_{B_0 C_0 D_0}$  are indeed in the form of (3.28) and (3.29), and obtain explicit expressions of  $F^{A_1}_{B_0 C_0 D_0}$ ,  $D^{A_{-1} A_0}_{B_0 C_0 D_0}$  and  $M^{C_{-1} B_{-1} A_{-1}}_{B_0 C_0 D_0}$ . Details of the calculation of  $A^{A_0}$  and  $B^{B_{-1} A_{-1}}$  are given in Appendix B, and we find

$$M^{C_{-1} B_{-1} A_{-1}}_{B_0 C_0 D_0} = 0. \quad (3.30)$$

and the following properties:

$$F^{A_1} \text{ does not depend on } \psi_\mu^\alpha \text{ and } c_{ab}. \quad (3.31)$$

$$D^{(\psi)(L)}_{(S)(S)(S)} \text{ is the only nonzero component} \\ \text{and it depends only on } e_\mu^a. \quad (3.32)$$

See (B.7), and (B.8) or (B.9) for explicit expressions. These will be used in the next section.

## 4 Constructing a master action

In this section we construct a master action  $S$  satisfying the classical master equation. Following the general theory of field-antifield formalism,

$$(C^A) = (C^{A_{-1}}, C^{A_0}, C^{A_1}, C^{A_2}), \quad (4.1)$$

are called fields, and we introduce corresponding antifields

$$(C_A^*) = (C_{A_{-1}}^*, C_{A_0}^*, C_{A_1}^*, C_{A_2}^*). \quad (4.2)$$

Statistical parity st, ghost number gh, and antighost number ag of these fields are

$$\text{st}[C^{A_n}] = A_n + n + 1, \quad \text{st}[C_{A_n}^*] = A_n + n, \quad (4.3)$$

$$\text{gh}[C^{A_n}] = n + 1, \quad \text{gh}[C_{A_n}^*] = -n - 2, \quad (4.4)$$

$$\text{ag}[C^{A_n}] = 0, \quad \text{ag}[C_{A_n}^*] = n + 2. \quad (4.5)$$

The importance of antighost number in order-by-order analysis of master actions has been pointed out in [15]. The antibracket  $(X, Y)$  is defined as

$$(X, Y) := \partial X / \partial C^A \cdot (\partial / \partial C_A^*) Y - \partial X / \partial C_A^* \cdot (\partial / \partial C^A) Y, \quad (4.6)$$

If  $X$  is Grassmann even i.e.  $\text{st}[X] = 0 \bmod 2$ , its ‘self-antibracket’ is

$$\frac{1}{2}(X, X) = \partial X / \partial C^A \cdot (\partial / \partial C_A^*) X. \quad (4.7)$$

Starting from the original action  $S_0 = \frac{1}{2\kappa^2} \mathcal{S}_0[C^{A-1}]$ , we add new terms which contain antifields, and the total action  $\mathcal{S} = \mathcal{S}_0 + \dots$  must satisfy the classical master equation:

$$(\mathcal{S}, \mathcal{S}) = 0. \quad (4.8)$$

To obtain a proper solution to this equation, terms consisting of one antifield and one ghost must be given by

$$\mathcal{S}_1 = C_{A-1}^* R^{A-1}{}_{A_0} C^{A_0} + C_{A_0}^* R^{A_0}{}_{A_1} C^{A_1} + C_{A_1}^* R^{A_1}{}_{A_2} C^{A_2}. \quad (4.9)$$

To see what terms we should add next, let us compute the self-antibracket of  $\mathcal{S}_0 + \mathcal{S}_1$ :

$$\begin{aligned} \frac{1}{2}(\mathcal{S}_0 + \mathcal{S}_1, \mathcal{S}_0 + \mathcal{S}_1) &= (-1)^{1+B_0} C_{B-1}^* \partial R^{B-1}{}_{B_0} / \partial C^{A-1} R^{A-1}{}_{A_0} C^{A_0} C^{B_0} \\ &= \frac{1}{2}(-1)^{1+B_0} C_{A-1}^* [R^{A-1}{}_{A_0} T^{A_0}{}_{B_0 C_0} \\ &\quad + \partial S_0 / \partial C^{B-1} E^{B-1 A-1}{}_{B_0 C_0}] C^{C_0} C^{B_0}, \end{aligned} \quad (4.10)$$

where we used (3.12). Our procedure to find new terms to be added is the following: Suppose we have the following expression in the result of the computation of the self-antibracket:

$$C_{C_{n_1}}^* C_{D_{n_2}}^* \dots \times C_{A_n}^* R^{A_n}{}_{B_{n+1}} K^{B_{n+1} C_{n_1} D_{n_2} \dots}{}_{C'_{m_1} D'_{m_2} \dots} [C^{A-1}] \times C^{C'_{m_1}} C^{D'_{m_2}} \dots, \quad (4.11)$$

then we add

$$\begin{aligned} &(-1)^{1+(C_{n_1}+n_1)+(D_{n_2}+n_2)+\dots} C_{C_{n_1}}^* C_{D_{n_2}}^* \dots \\ &\quad \times C_{A_{n+1}}^* K^{A_{n+1} C_{n_1} D_{n_2} \dots}{}_{C'_{m_1} D'_{m_2} \dots} [C^{A-1}] \times C^{C'_{m_1}} C^{D'_{m_2}} \dots, \end{aligned} \quad (4.12)$$

to the action. When indices have graded symmetry we need to put a combinatorial factor to the above. If we have the following expression

$$C_{C_{n_1}}^* C_{D_{n_2}}^* \dots \partial S_0 / \partial C^{A-1} K^{A-1 C_{n_1} D_{n_2} \dots}{}_{C'_{m_1} D'_{m_2} \dots} [C^{A-1}] \times C^{C'_{m_1}} C^{D'_{m_2}} \dots, \quad (4.13)$$

then we add

$$\begin{aligned} &(-1)^{1+(C_{n_1}+n_1)+(D_{n_2}+n_2)+\dots} C_{C_{n_1}}^* C_{D_{n_2}}^* \dots \\ &\quad \times C_{A-1}^* K^{A-1 C_{n_1} D_{n_2} \dots}{}_{C'_{m_1} D'_{m_2} \dots} [C^{A-1}] \times C^{C'_{m_1}} C^{D'_{m_2}} \dots, \end{aligned} \quad (4.14)$$

to the action. Contributions to the self-antibracket from these new terms cancel (4.11) and (4.13), and generate additional contribution. If it contains terms in the form of (4.11) and (4.13) again then we can repeat this procedure. As can be seen below, this procedure generates antighost number expansion of the action.

Let us apply this procedure to (4.10), which has antighost number 1. We obtain the following new terms of antighost number 2:

$$\mathcal{S}_{(0;0,0)} = \frac{1}{2}(-1)^{B_0} C_{A_0}^* T^{A_0}_{B_0 C_0} C^{C_0} C^{B_0}, \quad (4.15)$$

$$\mathcal{S}_{(-1,-1;0,0)} = \frac{1}{4}(-1)^{1+A_{-1}+B_0} C_{A_{-1}}^* C_{B_{-1}}^* E^{B_{-1}A_{-1}}_{B_0 C_0} C^{C_0} C^{B_0}, \quad (4.16)$$

and the self-antibracket of  $\mathcal{S}_{(2)} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_{(0;0,0)} + \mathcal{S}_{(-1,-1;0,0)}$  is

$$\begin{aligned} \frac{1}{2}(\mathcal{S}_{(2)}, \mathcal{S}_{(2)}) &= \frac{1}{2}(-1)^{C_0} C_{A_0}^* A^{A_0}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad - \frac{1}{4}(-1)^{A_{-1}+C_0} C_{A_{-1}}^* C_{B_{-1}}^* B^{B_{-1}A_{-1}}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad - \frac{1}{4}(-1)^{B_0+D_0+B_{-1}(B_0+C_0+A_{-1})} C_{B_{-1}}^* C_{A_0}^* \\ &\quad \times \partial T^{A_0}_{B_0 C_0} / \partial C^{A_{-1}} E^{B_{-1}A_{-1}}_{D_0 E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad - C_{A_0}^* T^{A_0}_{B_0 C_0} R^{C_0}_{A_1} C^{A_1} C^{B_0}, \end{aligned} \quad (4.17)$$

where we dropped terms which we can easily see vanish from (3.22). We also see that the last term in the above vanishes because  $R^{C_0}_{A_1} C^{A_1}$  is the gauge transformation of  $C^{C_0=[\mu\nu]}$ , and  $T^{A_0}_{B_0 C_0}$  contains  $C^{[\mu\nu]}$  only in the form of its field strength. Then using (3.28) and (3.29),

$$\begin{aligned} \frac{1}{2}(\mathcal{S}_{(2)}, \mathcal{S}_{(2)}) &= \frac{1}{2}(-1)^{C_0} C_{A_0}^* R^{A_0}_{A_1} F^{A_1}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad + \frac{1}{2}(-1)^{C_0} C_{A_0}^* \partial S_0 / \partial C^{B_{-1}} D^{B_{-1}A_0}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad - \frac{1}{2}(-1)^{A_{-1}+C_0+A_{-1}A_0} C_{A_{-1}}^* C_{B_{-1}}^* \\ &\quad \times R^{B_{-1}}_{A_0} D^{A_{-1}A_0}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad - \frac{1}{4}(-1)^{B_0+D_0+B_{-1}(B_0+C_0+A_{-1})} C_{B_{-1}}^* C_{A_0}^* \\ &\quad \times \partial T^{A_0}_{B_0 C_0} / \partial C^{A_{-1}} E^{B_{-1}A_{-1}}_{D_0 E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0}. \end{aligned} \quad (4.18)$$

The last term in the above is of antighost number 3, and the rest are of antighost number 2. Let us cancel the terms of lower antighost number: To cancel the first term in the above, we introduce the following term:

$$\mathcal{S}_{(1;0,0,0)} = \frac{1}{2}(-1)^{1+C_0} C_{A_1}^* F^{A_1}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0}. \quad (4.19)$$

To cancel the second and third term in the above, we introduce

$$\mathcal{S}_{(-1,0;0,0,0)} = \frac{1}{2}(-1)^{1+A_0+C_0} C_{A_0}^* C_{B_{-1}}^* D^{B_{-1}A_0}_{B_0C_0D_0} C^{D_0} C^{C_0} C^{B_0}. \quad (4.20)$$

The self-antibracket of  $\mathcal{S}_{(3)} = \mathcal{S}_{(2)} + \mathcal{S}_{(1;0,0,0)} + \mathcal{S}_{(-1,0;0,0,0)}$  is

$$\begin{aligned} \frac{1}{2}(\mathcal{S}_{(3)}, \mathcal{S}_{(3)}) &= \frac{1}{4}(-1)^{1+D_0+F_0+B_0(C_0+D_0)} C_{B_0}^* C_{A_0}^* \\ &\quad \times \partial T^{A_0}_{C_0D_0} / \partial C^{A_{-1}} D^{A_{-1}B_0}_{E_0F_0G_0} C^{G_0} C^{F_0} C^{E_0} C^{D_0} C^{C_0} \\ &\quad + \frac{1}{2}(-1)^{B_0+D_0} C_{A_1}^* Z^{A_1}_{B_0C_0D_0E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad + \frac{1}{2}(-1)^{A_0+B_0+D_0} C_{A_0}^* C_{B_{-1}}^* W^{B_{-1}A_0}_{B_0C_0D_0E_0} \\ &\quad \times C^{E_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad + \frac{3}{2}(-1)^{1+C_0} C_{A_1}^* F^{A_1}_{B_0C_0D_0} R^{D_0}_{B_1} C^{B_1} C^{C_0} C^{B_0}, \end{aligned} \quad (4.21)$$

where we dropped terms which we can easily see vanish from (3.22), (3.31) and (3.32).

$Z^{A_1}_{B_0C_0D_0E_0}$  and  $W^{B_{-1}A_0}_{B_0C_0D_0E_0}$  are defined as

$$Z^{A_1}_{B_0C_0D_0E_0} := \partial F^{A_1}_{[B_0C_0D_0]} / \partial C^{A_{-1}} R^{A_{-1}}_{E_0\} - \frac{3}{2} F^{A_1}_{[B_0C_0|F_0]} T^{F_0}_{D_0E_0\}, \quad (4.22)$$

$$\begin{aligned} W^{B_{-1}A_0}_{B_0C_0D_0E_0} &:= \partial D^{B_{-1}A_0}_{[B_0C_0D_0]} / \partial C^{A_{-1}} R^{A_{-1}}_{E_0\} \\ &\quad - \frac{3}{2} D^{B_{-1}A_0}_{[B_0C_0|F_0]} T^{F_0}_{D_0E_0\} \\ &\quad + (-1)^{B_{-1}(A_0+B_0+F_0)} T^{A_0}_{[B_0|F_0]} D^{B_{-1}F_0}_{C_0D_0E_0\} \\ &\quad + (-1)^{A_0B_0} \partial R^{B_{-1}}_{[B_0]} / \partial C^{A_{-1}} D^{A_{-1}A_0}_{C_0D_0E_0\} \\ &\quad + \frac{1}{2}(-1)^{1+B_{-1}(A_0+B_0+C_0+A_{-1})} \partial T^{A_0}_{[B_0C_0]} / \partial C^{A_{-1}} E^{B_{-1}A_{-1}}_{D_0E_0\}. \end{aligned} \quad (4.23)$$

The last term in (4.21) vanishes, because  $R^{D_0}_{B_1} C^{B_1}$  is the gauge transformation of  $C^{D_0=[\mu\nu]}$ , and  $F^{A_1}_{B_0C_0D_0}$  contains  $C^{[\mu\nu]}$  only in the form of its field strength.

In Appendix C, we show that

$$Z^{A_1}_{B_0C_0D_0E_0} = R^{A_1}_{A_2} Y^{A_2}_{B_0C_0D_0E_0}, \quad (4.24)$$

$$W^{B_{-1}A_0}_{C_0D_0E_0F_0} = R^{B_{-1}}_{B_0} V^{B_0A_0}_{C_0D_0E_0F_0}, \quad (4.25)$$

and explicit expressions of  $Y^{A_2}_{C_0D_0E_0F_0}$  and  $V^{B_0A_0}_{C_0D_0E_0F_0}$  are given by (C.4), and (C.9) or (C.10).  $Y^{A_2}_{C_0D_0E_0F_0}$  and  $V^{B_0A_0}_{C_0D_0E_0F_0}$  have graded antisymmetry in  $(C_0, D_0, E_0, F_0)$ , and graded symmetry in  $(A_0, B_0)$ .  $Y^{A_2}$  and  $V^{B_0A_0}$  are defined as

$$Y^{A_2} := (-1)^{B_0+D_0} Y^{A_2}_{B_0C_0D_0E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0}, \quad (4.26)$$

$$V^{B_0 A_0} := (-1)^{C_0 + E_0} V^{B_0 A_0}_{C_0 D_0 E_0 F_0} C^{F_0} C^{E_0} C^{D_0} C^{C_0}. \quad (4.27)$$

$Y^{A_2}$  and  $V^{B_0 A_0}_{C_0 D_0 E_0 F_0}$  have the following properties:

$$Y^{A_2} \text{ does not depend on } \psi_\mu^\alpha \text{ and } c_{[ab]}. \quad (4.28)$$

$$V^{(L)(L)}_{(S)(S)(S)(S)} \text{ is the only nonzero component} \\ \text{and it depends only on } e_\mu^a. \quad (4.29)$$

Then,

$$\begin{aligned} \frac{1}{2}(\mathcal{S}_{(3)}, \mathcal{S}_{(3)}) &= \frac{1}{4}(-1)^{1+D_0+F_0+B_0(C_0+D_0)} C_{B_0}^* C_{A_0}^* \\ &\quad \times \partial T^{A_0}_{C_0 D_0} / \partial C^{A-1}_{D_0} D^{A-1 B_0}_{E_0 F_0 G_0} C^{G_0} C^{F_0} C^{E_0} C^{D_0} C^{C_0} \\ &\quad + \frac{1}{2}(-1)^{B_0+D_0} C_{A_1}^* R^{A_1}_{A_2} Y^{A_2}_{B_0 C_0 D_0 E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad + \frac{1}{2}(-1)^{A_0+C_0+E_0} C_{A_0}^* C_{B_{-1}}^* R^{B_{-1}}_{B_0} V^{B_0 A_0}_{C_0 D_0 E_0 F_0} \\ &\quad \times C^{F_0} C^{E_0} C^{D_0} C^{C_0}. \end{aligned} \quad (4.30)$$

The first term in the above is of antighost number 4, and the rest are of antighost number 3. The terms of lower antighost number are canceled by introducing

$$\mathcal{S}_{(2;0,0,0,0)} = \frac{1}{2}(-1)^{1+B_0+D_0} C_{A_2}^* Y^{A_2}_{B_0 C_0 D_0 E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0}, \quad (4.31)$$

$$\mathcal{S}_{(0,0;0,0,0,0)} = \frac{1}{4}(-1)^{1+C_0+E_0} C_{A_0}^* C_{B_0}^* V^{B_0 A_0}_{C_0 D_0 E_0 F_0} C^{F_0} C^{E_0} C^{D_0} C^{C_0}, \quad (4.32)$$

and the self-antibracket of  $\mathcal{S}_{(4)} = \mathcal{S}_{(3)} + \mathcal{S}_{(2;0,0,0,0)} + \mathcal{S}_{(0,0;0,0,0,0)}$  is

$$\begin{aligned} \frac{1}{2}(\mathcal{S}_{(4)}, \mathcal{S}_{(4)}) &= \frac{1}{2}(-1)^{1+C_0+E_0} C_{A_2}^* X^{A_2}_{B_0 C_0 D_0 E_0 F_0} C^{F_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0} \\ &\quad + \frac{1}{4}(-1)^{1+D_0+F_0} C_{A_0}^* C_{B_0}^* U^{B_0 A_0}_{C_0 D_0 E_0 F_0 G_0} C^{G_0} C^{F_0} C^{E_0} C^{D_0} C^{C_0}, \end{aligned} \quad (4.33)$$

where we dropped terms which we can easily see vanish from (3.22), (3.31), (3.32), (4.28), and (4.29).  $X^{A_2}_{B_0 C_0 D_0 E_0 F_0}$  and  $U^{B_0 A_0}_{C_0 D_0 E_0 F_0 G_0}$  are defined as

$$X^{A_2}_{B_0 C_0 D_0 E_0 F_0} := \partial Y^{A_2}_{[B_0 C_0 D_0 E_0]} / \partial C^{A-1}_{F_0} R^{A(-1)}_{F_0} - 2Y^{A_2}_{[B_0 C_0 D_0 | G_0]} T^{G_0}_{E_0 F_0}, \quad (4.34)$$

$$\begin{aligned} U^{B_0 A_0}_{C_0 D_0 E_0 F_0 G_0} &:= \partial V^{B_0 A_0}_{[C_0 D_0 E_0 F_0]} / \partial C^{A-1}_{G_0} R^{A-1}_{G_0} \\ &\quad - 2V^{B_0 A_0}_{[C_0 D_0 E_0 | H_0]} T^{H_0}_{F_0 G_0} \end{aligned}$$

$$\begin{aligned}
& -(-1)^{A_0 C_0} T^{B_0}_{[C_0|H_0|} V^{H_0 A_0}_{D_0 E_0 F_0 G_0\}} \\
& -(-1)^{A_0 B_0 + B_0 C_0} T^{A_0}_{[C_0|H_0|} V^{H_0 B_0}_{D_0 E_0 F_0 G_0\}} \\
& + \frac{1}{2} (-1)^{A_0(C_0 + D_0)} \partial T^{B_0}_{[C_0 D_0]} / \partial C^{A-1} D^{A-1 A_0}_{E_0 F_0 G_0\}} \\
& + \frac{1}{2} (-1)^{A_0 B_0 + B_0(C_0 + D_0)} \partial T^{A_0}_{[C_0 D_0]} / \partial C^{A-1} D^{A-1 B_0}_{E_0 F_0 G_0\}}. \quad (4.35)
\end{aligned}$$

In Appendix D, we show  $X^{A_2}_{B_0 C_0 D_0 E_0 F_0} = 0$  and  $U^{B_0 A_0}_{C_0 D_0 E_0 F_0 G_0} = 0$ , which means that  $\mathcal{S}_{(4)}$  is a solution to the master equation  $\mathcal{S}$ :

$$\mathcal{S} = \mathcal{S}_{(4)}. \quad (4.36)$$

## 5 Conclusion

We have constructed an explicit expression of the master action of  $D = 11$  supergravity. For readers' convenience we summarize our result: A solution to the classical master equation  $\mathcal{S}$  is

$$\begin{aligned}
2\kappa^2 \mathcal{S} = & \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_{(0;0,0)} + \mathcal{S}_{(-1,-1;0,0)} \\
& + \mathcal{S}_{(1;0,0,0)} + \mathcal{S}_{(-1,0;0,0,0)} + \mathcal{S}_{(2;0,0,0,0)} + \mathcal{S}_{(0,0;0,0,0,0)}, \quad (5.1)
\end{aligned}$$

where  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ ,  $\mathcal{S}_{(0;0,0)}$ ,  $\mathcal{S}_{(-1,-1;0,0)}$ ,  $\mathcal{S}_{(1;0,0,0)}$ ,  $\mathcal{S}_{(-1,0;0,0,0)}$ ,  $\mathcal{S}_{(2;0,0,0,0)}$ ,  $\mathcal{S}_{(0,0;0,0,0,0)}$  are given by (2.1), (4.9), (4.15), (4.16), (4.19), (4.20), (4.32), and (4.32) respectively, and symbols used in them are defined by (3.16)-(3.19), (3.21) (B.7), (B.8), (C.4), (C.9) (or, (B.9) and (C.10)).

Note that the master action has an ambiguity that terms in the form of canonical transformation can be added. Up to this ambiguity our solution is essentially unique, as has been shown in [15] generally. To construct a gauge fixed action, we have to give a gauge fixing fermion, and introduce more field-antifield pairs and more terms to the action, but it is completely straightforward. (See e.g. [11].)

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## Appendix

## A Notation about spinors and Fierz transformation

In this appendix we summarize our notation about spinors, and explain about details on Fierz transformation.  $32 \times 32$  gamma matrices  $(\Gamma^a)^\alpha_\beta$  in eleven dimensions satisfy

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \text{diag}(-1, +1, \dots, +1), \quad (\text{A.1})$$

and the charge conjugation matrix  $\mathcal{C}^{\alpha\beta}$  is an antisymmetric matrix obeying  $\mathcal{C}^{-1}\Gamma^a\mathcal{C} = -(\Gamma^a)^T$ . 10th gamma matrix  $\Gamma^{10}$  is given by the product of other gamma matrices:

$$\Gamma^{10} = \sigma\Gamma^0\Gamma^1\dots\Gamma^9, \quad \sigma = \pm 1. \quad (\text{A.2})$$

A Majorana spinor  $\psi$  satisfies the following relation:

$$\bar{\psi} = \psi^\dagger\Gamma^0 = -\psi^T\mathcal{C}^{-1}. \quad (\text{A.3})$$

The totally antisymmetric tensor is defined as

$$\epsilon^{a_1\dots a_{11}} = \text{sgn}(a_1, \dots, a_{11}), \quad (\text{A.4})$$

where  $\text{sgn}(a_1, \dots, a_{11})$  is the sign of the permutation  $(a_1, \dots, a_{11})$ . Then

$$\epsilon^{\mu_1\dots\mu_{11}} = e^{-1}\text{sgn}(\mu_1, \dots, \mu_{11}). \quad (\text{A.5})$$

Gamma matrices have a ‘duality’ relation:

$$\Gamma_{a_1\dots a_n} = \frac{\sigma(-1)^{\frac{1}{2}(n+1)(n+2)}}{(11-n)!}\epsilon^{a_1\dots a_{11}}\Gamma_{a_{n+1}\dots a_{11}}, \quad (\text{A.6})$$

which means that  $\Gamma_{a_1\dots a_n}$  ( $n = 0, 1, \dots, 5$ ) are independent and higher gamma matrices can be expressed by the lower ones. Therefore we can take  $\mathcal{C}^{-1}\Gamma_{a_1\dots a_n}$  ( $n = 0, 1, \dots, 5$ ) as a basis of  $32 \times 32$  matrices. For  $n = 1, 2$ , and  $5$  these matrices are symmetric, and others are antisymmetric.

We often need to perform Fierz transformation. Especially we often need to show that sums of terms in the form of  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2$ ,  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 c$ , and  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 c \cdot \bar{c}\Gamma_3$  cancel, where  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$  are products of gamma matrices and  $c^\alpha$  is a Grassmann even spinor. For this purpose the followings can be used:

$$(\Gamma_1)^\alpha_{(\beta}(\mathcal{C}^{-1}\Gamma_2)_{\gamma\delta)} = \frac{1}{3}\left[(\Gamma_1)^\alpha_{\beta}(\mathcal{C}^{-1}\Gamma_2)_{\gamma\delta}\right]$$



$$+ \sum_{n=1,2,5} \frac{(-1)^{n(n-1)/2}}{16n!} (\Gamma_1 \Gamma^{a_1 \dots a_n} \Gamma_2)_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_{a_1 \dots a_n})_{\gamma\delta}, \quad (\text{A.7})$$

$$(\mathcal{C}^{-1} \Gamma_1)_{(\alpha\beta} (\mathcal{C}^{-1} \Gamma_2)_{\gamma\delta)} = \frac{1}{6} \left[ (\mathcal{C}^{-1} \Gamma_1)_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_2)_{\gamma\delta} + (\mathcal{C}^{-1} \Gamma_2)_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_1)_{\gamma\delta} \right. \\ \left. + \sum_{n=1,2,5} \frac{(-1)^{n(n-1)/2}}{16n!} \{ (\mathcal{C}^{-1} \Gamma_1 \Gamma^{a_1 \dots a_n} \Gamma_2)_{(\alpha\beta)} + (\mathcal{C}^{-1} \Gamma_2 \Gamma^{a_1 \dots a_n} \Gamma_1)_{(\alpha\beta)} \} (\mathcal{C}^{-1} \Gamma_{a_1 \dots a_n})_{\gamma\delta} \right], \quad (\text{A.8})$$

$$(\mathcal{C}^{-1} \Gamma_1)_{(\alpha\beta} (\mathcal{C}^{-1} \Gamma_2)_{\gamma\delta} (\mathcal{C}^{-1} \Gamma_3)_{\epsilon\zeta)} = \frac{1}{5!} \left[ 4 (\mathcal{C}^{-1} \Gamma_1)_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_2)_{\gamma\delta} (\mathcal{C}^{-1} \Gamma_3)_{\epsilon\zeta} \right. \\ + \frac{1}{8} \sum_{n=1,2,5} \frac{(-1)^{n(n-1)/2}}{n!} (\mathcal{C}^{-1} \Gamma_1 \Gamma^{a_1 \dots a_n} \Gamma_2)_{(\alpha\beta)} (\mathcal{C}^{-1} \Gamma^{a_1 \dots a_n})_{\gamma\delta} (\mathcal{C}^{-1} \Gamma_3)_{\epsilon\zeta} \\ + \frac{1}{8} \sum_{n=1,2,5} \frac{(-1)^{n(n-1)/2}}{n!} (\mathcal{C}^{-1} \Gamma^{a_1 \dots a_n})_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_1 \Gamma^{a_1 \dots a_n} \Gamma_2)_{(\gamma\delta)} (\mathcal{C}^{-1} \Gamma_3)_{\epsilon\zeta} \\ + \frac{1}{4} \sum_{n=1,2,5} \frac{(-1)^{n(n-1)/2}}{n!} (\mathcal{C}^{-1} \Gamma_1)_{\alpha\beta} (\mathcal{C}^{-1} \Gamma^{a_1 \dots a_n})_{\gamma\delta} (\mathcal{C}^{-1} \Gamma_2 \Gamma^{a_1 \dots a_n} \Gamma_3)_{\epsilon\zeta} \\ + \frac{1}{4} \sum_{n=1,2,5} \frac{(-1)^{n(n-1)/2}}{n!} (\mathcal{C}^{-1} \Gamma^{a_1 \dots a_n})_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_1)_{\gamma\delta} (\mathcal{C}^{-1} \Gamma_2 \Gamma^{a_1 \dots a_n} \Gamma_3)_{\epsilon\zeta} \\ + \frac{1}{64} \sum_{n=1,2,5} \sum_{m=1,2,5} \frac{(-1)^{n(n-1)/2}}{n!} \frac{(-1)^{m(m-1)/2}}{m!} \\ \times (\mathcal{C}^{-1} \Gamma_1 \Gamma^{b_1 \dots b_m} \Gamma_{a_1 \dots a_n})_{(\alpha\beta)} (\mathcal{C}^{-1} \Gamma^{b_1 \dots b_m})_{\gamma\delta} (\mathcal{C}^{-1} \Gamma_2 \Gamma^{a_1 \dots a_n} \Gamma_3)_{\epsilon\zeta} \\ + \frac{1}{64} \sum_{n=1,2,5} \sum_{m=1,2,5} \frac{(-1)^{n(n-1)/2}}{n!} \frac{(-1)^{m(m-1)/2}}{m!} \\ \times (\mathcal{C}^{-1} \Gamma^{b_1 \dots b_m})_{\alpha\beta} (\mathcal{C}^{-1} \Gamma_1 \Gamma^{b_1 \dots b_m} \Gamma_{a_1 \dots a_n})_{(\gamma\delta)} (\mathcal{C}^{-1} \Gamma_2 \Gamma^{a_1 \dots a_n} \Gamma_3)_{\epsilon\zeta} \\ \left. \right] + (\Gamma_1 \leftrightarrow \Gamma_2). \quad (\text{A.9})$$

Applying these transformations we obtain, for example, the following Fierz identities:

$$0 = (\Gamma^b \mathcal{C})^{(\alpha\beta} (\Gamma_{ab} \mathcal{C})^{\gamma\delta)}, \quad (\text{A.10})$$

$$0 = (\Gamma^b \mathcal{C})^{(\alpha\beta} (\Gamma_{ba_1 \dots a_4} \mathcal{C})^{\gamma\delta)} - 3 (\Gamma_{[a_1 a_2} \mathcal{C})^{(\alpha\beta} (\Gamma_{a_3 a_4]} \mathcal{C})^{\gamma\delta)}, \quad (\text{A.11})$$

$$0 = (\Gamma^{bc} \mathcal{C})^{\alpha(\beta} (\Gamma_{bca_1 \dots a_4} \mathcal{C})^{\gamma\delta)} - 2 (\Gamma_{a_1 \dots a_4 b} \mathcal{C})^{\alpha(\beta} (\Gamma^b \mathcal{C})^{\gamma\delta)} \\ - 16 (\Gamma_{[a_1 a_2 a_3} \mathcal{C})^{\alpha(\beta} (\Gamma_{a_4]} \mathcal{C})^{\gamma\delta)} + 24 (\Gamma_{[a_1 a_2} \mathcal{C})^{\alpha(\beta} (\Gamma_{a_3 a_4]} \mathcal{C})^{\gamma\delta)}, \quad (\text{A.12})$$

$$0 = (\Gamma_{c_1 c_2} \mathcal{C})^{\alpha(\beta} (\Gamma^{ba_1 a_2 a_3 c_1 c_2} \mathcal{C})^{\gamma\delta)} - (\Gamma^{ba_1 a_2 a_3 c_1 c_2} \mathcal{C})^{\alpha(\beta} (\Gamma_{c_1 c_2} \mathcal{C})^{\gamma\delta)} \\ + 2 (\Gamma_c \mathcal{C})^{\alpha(\beta} (\Gamma^{ba_1 a_2 a_3 c} \mathcal{C})^{\gamma\delta)} - 2 (\Gamma^{ba_1 a_2 a_3 c} \mathcal{C})^{\alpha(\beta} (\Gamma_c \mathcal{C})^{\gamma\delta)}$$

$$+8(\Gamma^{a_1 a_2 a_3} \mathcal{C})^{\alpha(\beta} (\Gamma^b \mathcal{C})^{\gamma\delta)} - 24(\Gamma^{b[a_1 a_2} \mathcal{C})^{\alpha(\beta} (\Gamma^{a_3]} \mathcal{C})^{\gamma\delta)}. \quad (\text{A.13})$$

The first one in the above is well-known: it ensures M2-brane kappa symmetry. However, rather than to apply (A.7), (A.8), and (A.9) directly, it is easier to apply the following procedure: From (A.7) we obtain

$$(\Gamma^b \mathcal{C})^{\alpha(\beta} (\Gamma_{ab} \mathcal{C})^{\gamma\delta)} = -(\Gamma_a \Gamma^b \mathcal{C})^{\alpha(\beta} (\Gamma_b \mathcal{C})^{\gamma\delta)} + (\mathcal{C})^{\alpha(\beta} (\Gamma_a \mathcal{C})^{\gamma\delta)}, \quad (\text{A.14})$$

$$\begin{aligned} (\Gamma^b \mathcal{C})^{\alpha(\beta} (\Gamma_{a_1 \dots a_4 b} \mathcal{C})^{\gamma\delta)} &= 6(\Gamma_{[a_1 a_2} \mathcal{C})^{\alpha(\beta} (\Gamma_{a_3 a_4]} \mathcal{C})^{\gamma\delta)} - (\Gamma_{a_1 \dots a_4} \Gamma^b \mathcal{C})^{\alpha(\beta} (\Gamma_b \mathcal{C})^{\gamma\delta)} \\ &\quad + 4(\Gamma_{[a_1 a_2 a_3} \mathcal{C})^{\alpha(\beta} (\Gamma_{a_4]} \mathcal{C})^{\gamma\delta)}. \end{aligned} \quad (\text{A.15})$$

Note that in the above relations the number of local Lorentz indices of gamma matrices with spinor indices symmetrized on the right hand sides are smaller than those on the left hand sides. Therefore we can use them for reducing the numbers of local Lorentz indices of gamma matrices sandwiched by  $c$ , if they are equal to 2 or 5 and some of the indices are contracted. If the gamma matrices have more than 5 indices, we can reduce the number by the following double duality relation without totally antisymmetric tensors: For  $n \leq m$ ,

$$\begin{aligned} (\Gamma_{a_1 \dots a_n}^{c_1 \dots c_l})^\alpha_\beta (\Gamma^{b_1 \dots b_m}_{c_1 \dots c_l})^\gamma_\delta &= (-1)^{1+n(n-1)/2+m(m-1)/2} \frac{l!(11-l)!}{(11-n-l)!(11-m-l)!} \\ &\quad \times \delta_{a_1}^{[b_1} \dots \delta_{a_n}^{b_n} \delta_{d_1}^{b_{n+1}} \dots \delta_{d_{m-n}}^{b_m} \delta_{d_{m-n+1}}^{e_1} \dots \delta_{d_{11-n-l}}^{e_{11-m-l}} \\ &\quad \times (\Gamma^{d_1 \dots d_{11-n-l}})^\alpha_\beta (\Gamma_{e_1 \dots e_{11-m-l}})^\gamma_\delta. \end{aligned} \quad (\text{A.16})$$

By applying these repeatedly for reducing the numbers of indices as much as possible, we see the cancellation of terms more easily. In Mathematica calculations, especially for terms in the form of  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 c \cdot \bar{c}\Gamma_3$ , this procedure gives an algorithm much faster than using (A.7), (A.8), and (A.9) directly.

(A.16) also means that there exist some relations between  $\bar{c}\Gamma_{a_1 \dots a_n}^{c_1 \dots c_l} c \bar{c}\Gamma^{b_1 \dots b_m}_{c_1 \dots c_l} c$  for  $n + m + 2l = 11$ . They are given by

$$\begin{aligned} 0 &= \bar{c}\Gamma_{c_1 \dots c_5} c \bar{c}\Gamma^{b c_1 \dots c_5} c, \\ 0 &= \bar{c}\Gamma_{a_1 a_2 c_1 \dots c_3} c \bar{c}\Gamma^{b_1 b_2 b_3 c_1 \dots c_3} c + \frac{3}{4} \delta_{[a_1}^{[b_1} \bar{c}\Gamma_{a_2] c_1 \dots c_4} c \bar{c}\Gamma^{b_1 b_2] c_1 \dots c_4} c, \\ 0 &= \bar{c}\Gamma_{a_1 a_2 a_3 a_4 c_1} c \bar{c}\Gamma^{b_1 b_2 b_3 b_4 b_5 c_1} c + 5 \delta_{[a_1}^{[b_1} \bar{c}\Gamma_{a_2 a_3 a_4] c_1 c_2} c \bar{c}\Gamma^{b_2 b_3 b_4 b_5] c_1 c_2} c \\ &\quad - \frac{5}{2} \delta_{[a_1}^{[b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \bar{c}\Gamma_{a_4] c_1 \dots c_4} c \bar{c}\Gamma^{b_4 b_5] c_1 \dots c_4} c, \\ 0 &= \bar{c}\Gamma_c c \bar{c}\Gamma^{b_1 \dots b_9} c, \\ 0 &= \bar{c}\Gamma_{a_1 c} c \bar{c}\Gamma^{b_1 \dots b_8 c} c + 2 \delta_{a_1}^{[b_1} \bar{c}\Gamma_{c_1 c_2} c \bar{c}\Gamma^{b_2 \dots b_8] c_1 c_2} c. \end{aligned} \quad (\text{A.17})$$

These also help us to see the cancellation of terms in the form of  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 c$ .

## B Details of $A$ and $B$

Let us calculate

$$A^{A_0} = (-1)^{C_0} A^{A_0}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0}, \quad (\text{B.1})$$

$$B^{B_{-1} A_{-1}} = (-1)^{C_0} B^{B_{-1} A_{-1}}_{B_0 C_0 D_0} C^{D_0} C^{C_0} C^{B_0}, \quad (\text{B.2})$$

by setting the symbols in the definition (3.25) and (3.26) to those given by (3.16), (3.17), (3.18), (3.19) and (3.21), and see if they are indeed in the form of (3.28) and (3.29).

First let us calculate  $A^{A_0}$ . Most of the calculation is straightforward, except that we need Fierz identity (A.10) for  $A_0 = (A)$  and (A.12) for  $A_0 = (S)$ . The result is

$$A^{[\mu\nu]} = R^{[\mu\nu]}_{A_1} F^{A_1} = \partial_\mu F^{[\nu]} - \partial_\nu F^{[\mu]}, \quad (\text{B.3})$$

$$A^{[ab]} = \partial \mathcal{S}_0 / \partial \psi_\mu^\alpha D^{(\mu\alpha)[ab]} = -ie D^{(\mu\alpha)[ab]} (\mathcal{C}^{-1} \Gamma^{\mu\nu\lambda} \tilde{D}_\nu \psi_\lambda)_\alpha, \quad (\text{B.4})$$

$$A^{(D)} = 0, \quad (\text{B.5})$$

$$A^{(S)} = 0, \quad (\text{B.6})$$

where

$$F^{[\mu]} = 3c^\nu c^\lambda \partial_{[\mu} c_{\nu\lambda]} - \frac{i}{4} c^\nu \bar{c} \Gamma^\lambda c A_{\mu\nu\lambda} - \frac{i}{4} c^\nu \bar{c} \Gamma_{\mu\nu} c, \quad (\text{B.7})$$

$$\begin{aligned} D^{(\mu\alpha)[ab]} = & -\frac{i}{12} \frac{1}{576} e^{-1} \left[ \frac{17}{2} \bar{c} \Gamma^c c \bar{c} \Gamma^{ab} c_\mu \mathcal{C} - \frac{61}{2} \bar{c} \Gamma_\mu c \bar{c} \Gamma^{ab} \mathcal{C} + \bar{c} \Gamma^{[a} c \bar{c} \Gamma^{b]}_\mu \mathcal{C} \right. \\ & - 31 e_\mu^{[a} \bar{c} \Gamma^{b]} c \bar{c} \mathcal{C} - 31 e_\mu^{[a} \bar{c} \Gamma_c c \bar{c} \Gamma^{b]} c \mathcal{C} \\ & - \frac{11}{4} \bar{c} \Gamma^{c_1 c_2} c \bar{c} \Gamma^{ab}_{c_1 c_2 \mu} \mathcal{C} + \frac{7}{2} \bar{c} \Gamma^c_\mu c \bar{c} \Gamma^{ab} c \mathcal{C} + \frac{19}{2} \bar{c} \Gamma^{ab} c \bar{c} \Gamma_\mu c \\ & + 5 \bar{c} \Gamma^{[a} c \bar{c} \Gamma^{b]} c_\mu \mathcal{C} - 97 \bar{c} \Gamma^{[a}_\mu c \bar{c} \Gamma^{b]} \mathcal{C} \\ & - \frac{19}{2} e_\mu^{[a} \bar{c} \Gamma_{c_1 c_2} c \bar{c} \Gamma^{b]} c_{c_1 c_2} \mathcal{C} + 17 e_\mu^{[a} \bar{c} \Gamma^{b]} c c \bar{c} \Gamma_c \mathcal{C} \\ & + \frac{1}{24} \bar{c} \Gamma^{[a}_{c_1 \dots c_4} c \bar{c} \Gamma^{b]} c_{c_1 \dots c_4} \mu \mathcal{C} - \frac{5}{6} \bar{c} \Gamma^{[a}_{c_1 c_2 c_3 \mu} c \bar{c} \Gamma^{b]} c_{c_1 c_2 c_3} \mathcal{C} \\ & + \frac{17}{240} \bar{c} \Gamma^{c_1 \dots c_5} c \bar{c} \Gamma^{ab}_{c_1 \dots c_5 \mu} \mathcal{C} - \frac{13}{48} \bar{c} \Gamma^{c_1 \dots c_4}_\mu c \bar{c} \Gamma^{ab}_{c_1 \dots c_4} \mathcal{C} \\ & - \frac{11}{4} \bar{c} \Gamma^{abc_1 c_2}_\mu c \bar{c} \Gamma_{c_1 c_2} \mathcal{C} + \frac{7}{12} \bar{c} \Gamma^{abc_1 c_2 c_3} c \bar{c} \Gamma_{c_1 c_2 c_3 \mu} \mathcal{C} \\ & \left. - \frac{7}{120} e_\mu^{[a} \bar{c} \Gamma^{c_1 \dots c_5} c \bar{c} \Gamma^{b]}_{c_1 \dots c_5} \mathcal{C} - \frac{7}{24} e_\mu^{[a} \bar{c} \Gamma^{b]} c_{c_1 \dots c_4} c \bar{c} \Gamma_{c_1 \dots c_4} \mathcal{C} \right]^\alpha. \quad (\text{B.8}) \end{aligned}$$

The ambiguity in the definition of  $F^{A_1}$  is fixed so that  $F^{A_1}$  contains  $c_{\mu\nu}$  only in the form of its field strength, and the ambiguity in the definition of  $D^{(\mu\alpha)[ab]}$  is fixed similarly to  $E^{B_{-1} A_{-1}}_{B_0 C_0}$ .

The expression (B.8) is already symmetrized under interchange of three  $c^\epsilon$ s. i.e.  $D^{(\mu\alpha)[ab]}_{(\beta)(\gamma)(\delta)}$  is given just by removing  $c^\epsilon$  in (B.8), and putting indices  $\beta, \gamma$  and  $\delta$ :

$$D^{(\mu\alpha)[ab]}_{(\beta)(\gamma)(\delta)} = -\frac{i}{12} \frac{1}{576} e^{-1} \left[ \frac{17}{2} (\mathcal{C}^{-1} \Gamma^c)_{\beta\gamma} (\mathcal{C}^{-1} \Gamma^{ab} c_\mu \mathcal{C})_\delta^\alpha - \frac{61}{2} (\mathcal{C}^{-1} \Gamma_\mu)_{\beta\gamma} (\mathcal{C}^{-1} \Gamma^{ab} \mathcal{C})_\delta^\alpha + \dots \right]. \quad (\text{B.9})$$

This can be seen from the fact that (B.8) is invariant if we apply (A.7). However the following reduced expression, which is given by applying (A.14), (A.15), and (A.16) to (B.8), is simpler:

$$D^{(\mu\alpha)[ab]} = -\frac{i}{288} e^{-1} \left[ -6e_\mu^{[a} \bar{c} \Gamma_c \bar{c} \Gamma^{b]c} \mathcal{C} - 3\bar{c} \Gamma_\mu c \bar{c} \Gamma^{ab} \mathcal{C} + 2\bar{c} \Gamma^{ab} c \bar{c} \Gamma_\mu \mathcal{C} + 12\bar{c} \Gamma_\mu^{[a} c \bar{c} \Gamma^{b]} \mathcal{C} \right]^\alpha. \quad (\text{B.10})$$

If we read off  $D^{(\mu\alpha)[ab]}$  from this expression we need explicit symmetrization of indices:

$$D^{(\mu\alpha)[ab]}_{(\beta)(\gamma)(\delta)} = -\frac{i}{288} e^{-1} \left[ -6e_\mu^{[a} (\mathcal{C}^{-1} \Gamma_c)_{(\beta\gamma)} (\mathcal{C}^{-1} \Gamma^{b]c} \mathcal{C})_\delta^\alpha - 3(\mathcal{C}^{-1} \Gamma_\mu)_{(\beta\gamma)} (\mathcal{C}^{-1} \Gamma^{ab} \mathcal{C})_\delta^\alpha + \dots \right]. \quad (\text{B.11})$$

Thus we see that  $D^{(\mu\alpha)[ab]}$  is the only nonzero component of  $D^{B_{-1}A_{-1}}$ .

It is an important check to confirm that we can obtain the same  $D^{B_{-1}A_{-1}}$  from  $B^{B_{-1}A_{-1}}$ . From (3.26) and (3.22), we see that  $B^{B_{-1}A_{-1}}$  does not vanish only if either  $B_{-1}$  or  $A_{-1}$  is  $(\psi)$ , and

$$B^{(\nu a)(\mu\alpha)} = -\frac{i}{4} (\bar{c} \Gamma^a)_\beta E^{(\nu\beta)(\mu\alpha)}, \quad (\text{B.12})$$

$$B^{(\rho\alpha)[\mu\nu\lambda]} = -\frac{3}{4} i (\bar{c} \Gamma_{[\mu\nu})_\beta E^{(\rho\alpha)(\lambda]\beta)}. \quad (\text{B.13})$$

Using (3.21), both of the above are expressed by sums of terms in the form of  $\bar{c} \Gamma_1 c (\bar{c} \Gamma_2 \mathcal{C})^\alpha$ . By performing Fierz transformation (A.7) (or the faster procedure) to them, we see that  $B^{(\rho\alpha)[\mu\nu\lambda]}$  vanishes, and  $B^{(\nu a)(\mu\alpha)}$  indeed gives the same  $D^{(\mu\alpha)[ab]}$  as (B.8):

$$B^{(\nu a)(\mu\alpha)} = e_{\nu b} D^{(\mu\alpha)[ab]}, \quad (\text{B.14})$$

$$B^{(\rho\alpha)[\mu\nu\lambda]} = 0. \quad (\text{B.15})$$

Then the final task is to calculate  $B^{(\nu\beta)(\mu\alpha)}$ . Because we need similar calculations in the following appendices, we explain the detail of the calculation in this case. From (3.26) and (3.22),

$$B^{(\nu\beta)(\mu\alpha)} = \partial E^{(\nu\beta)(\mu\alpha)} / \partial e_\lambda^a R^{(\lambda a)}_{A_0} C^{A_0} - E^{(\nu\beta)(\mu\alpha)}_{(\gamma)(\delta)} T^{(\delta)} c^\gamma + \partial(R^{(\nu\beta)}_{A_0} C^{A_0}) / \partial \psi_\lambda^\delta E^{(\lambda\delta)(\mu\alpha)} + \partial(R^{(\mu\alpha)}_{A_0} C^{A_0}) / \partial \psi_\lambda^\delta E^{(\lambda\delta)(\nu\beta)}. \quad (\text{B.16})$$

The first term in (B.16) can be calculated by making the symmetry transformation for  $e_\mu^a$  in  $E^{(\nu\beta)(\mu\alpha)}$  with transformation parameters replaced by  $C^{A_0}$ , which is denoted by  $\tilde{\delta}$ . This replacement is done after reordering the parameters to the rightmost position:

$$\begin{aligned} \tilde{\delta} e_\mu^a &= (\tilde{\delta}^S + \tilde{\delta}^D + \tilde{\delta}^L + \tilde{\delta}^A) e_\mu^a \\ &= -\frac{i}{4} \bar{c} \Gamma^a \psi_\mu - c^\nu \partial_\nu e_\mu^a - \partial_\mu c^\nu e_\nu^a + c^a_b e_\mu^b, \end{aligned} \quad (\text{B.17})$$

where the first term obtains an additional sign factor due to the reordering.  $\tilde{\delta} \psi_\mu^\alpha$  is defined similarly.

The second term in (B.16) can be calculated just by replacing one of  $c^\gamma$ s in  $E^{(\nu\beta)(\mu\alpha)}/2$  by  $T^{(\gamma)}$ :

$$\begin{aligned} -E^{(\nu\beta)(\mu\alpha)}_{(\gamma)(\delta)} T^{(\delta)} c^\gamma &= \frac{1}{2} \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma T^{(\gamma)} \\ &= \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma \left( \frac{1}{4} c_{ab} \Gamma^{ab} c \right)^\gamma - c^\lambda \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma \partial_\lambda c^\gamma - \frac{i}{8} \bar{c} \Gamma^\lambda c \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma \psi_\lambda^\gamma. \end{aligned} \quad (\text{B.18})$$

The third and fourth terms in (B.16) are rewritten as

$$\partial(\tilde{\delta} \psi_\nu^\beta) / \partial \psi_\lambda^\delta E^{(\lambda\delta)(\mu\alpha)} + ((\mu\alpha) \leftrightarrow (\nu\beta)). \quad (\text{B.19})$$

Let us calculate this for each of  $\tilde{\delta}^S$ ,  $\tilde{\delta}^D$ , and  $\tilde{\delta}^L$  in  $\tilde{\delta}$ :

$$\begin{aligned} \partial(\tilde{\delta}^L \psi_\nu^\beta) / \partial \psi_\lambda^\delta E^{(\lambda\delta)(\mu\alpha)} + ((\mu\alpha) \leftrightarrow (\nu\beta)) \\ = -\frac{1}{4} c_{ab} (\Gamma^{ab})^\beta_\gamma E^{(\nu\gamma)(\mu\alpha)} - \frac{1}{4} c_{ab} (\Gamma^{ab})^\alpha_\gamma E^{(\mu\gamma)(\nu\beta)}, \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned} \partial(\tilde{\delta}^S \psi_\nu^\beta) / \partial \psi_\lambda^\delta E^{(\lambda\delta)(\mu\alpha)} + ((\mu\alpha) \leftrightarrow (\nu\beta)) \\ = \frac{1}{4} (\Gamma^{bc} c)^\beta \partial \hat{\omega}_{\nu ab} / \partial \psi_\lambda^\gamma E^{(\lambda\gamma)(\mu\alpha)} + \frac{1}{288} [(\Gamma^{\mu_1 \dots \mu_4}{}_\nu - 8 \delta_\nu^{\mu_1} \Gamma^{\mu_2 \mu_3 \mu_4}) c]^\beta \partial \hat{F}_{\mu_1 \dots \mu_4} / \partial \psi_\lambda^\gamma E^{(\lambda\gamma)(\mu\alpha)} \\ + ((\mu\alpha) \leftrightarrow (\nu\beta)), \end{aligned} \quad (\text{B.21})$$

and we have to be careful with  $\tilde{\delta}^D$  because it contains derivative operators. Indices  $(\mu\alpha)$  and  $(\nu\beta)$  must be supplemented with spacetime positions:  $(\mu\alpha) \rightarrow (\mu\alpha x)$  and  $(\nu\beta) \rightarrow (\nu\beta y)$ . Then,

$$\begin{aligned}
& \partial(\tilde{\delta}^D \psi_\nu^\beta) / \partial \psi_\lambda^\delta E^{(\lambda\delta)(\mu\alpha)} + ((\mu\alpha) \leftrightarrow (\nu\beta)) \\
&= \partial_\nu c^\lambda(y) E^{(\lambda\beta)(\mu\alpha)}(y) \delta(y-x) + \int dz c^\lambda(y) \partial_\lambda^y \delta(y-z) E^{(\nu\beta)(\mu\alpha)}(z) \delta(z-x) \\
&\quad + ((\mu\alpha x) \leftrightarrow (\nu\beta y)) \\
&= (\partial_\nu c^\lambda(y) E^{(\lambda\beta)(\mu\alpha)}(y) + \partial_\mu c^\lambda(y) E^{(\nu\beta)(\lambda\alpha)}(y)) \delta(y-x) \\
&\quad + (c^\lambda(y) E^{(\nu\beta)(\mu\alpha)}(x) - c^\lambda(x) E^{(\nu\beta)(\mu\alpha)}(y)) \partial_\lambda^y \delta(y-x) \\
&= (\partial_\nu c^\lambda(y) E^{(\lambda\beta)(\mu\alpha)}(y) + \partial_\mu c^\lambda(y) E^{(\nu\beta)(\lambda\alpha)}(y) \\
&\quad - \partial_\lambda c^\lambda(y) E^{(\nu\beta)(\mu\alpha)}(y) + c^\lambda(y) \partial_\lambda E^{(\nu\beta)(\mu\alpha)}(y)) \delta(y-x). \quad (B.22)
\end{aligned}$$

Noting that  $E^{(\nu\beta)(\mu\alpha)}$  is in the following form,

$$E^{(\nu\beta)(\mu\alpha)} = e^{-1} e_\mu^a e_\nu^b \cdot (e_\lambda^c \text{-independent part}), \quad (B.23)$$

(B.22) is equal to minus the diffeomorphism transformation of  $E^{(\nu\beta)(\mu\alpha)}$  with the parameter replaced by  $c^\lambda$ . and it is equal to

$$- \tilde{\delta}^D E^{(\nu\beta)(\mu\alpha)} + c^\lambda \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma \partial_\lambda c^\gamma. \quad (B.24)$$

(Note that  $\tilde{\delta}^D$  does not act on  $c^\gamma$ .) Hence terms proportional to  $c^\lambda$  in (B.16) cancel.

Next let us collect terms proportional to  $c_{ab}$  in (B.16):

$$\begin{aligned}
& \tilde{\delta}^L E^{(\nu\beta)(\mu\alpha)} + \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma (\frac{1}{4} c_{ab} \Gamma^{ab} c)^\gamma \\
& - \frac{1}{4} c_{ab} (\Gamma^{ab})^\beta_\gamma E^{(\nu\gamma)(\mu\alpha)} - \frac{1}{4} c_{ab} (\Gamma^{ab})^\alpha_\gamma E^{(\nu\beta)(\mu\gamma)}. \quad (B.25)
\end{aligned}$$

We see that the above vanish again, because the first and second term give local Lorentz transformation of  $E^{(\nu\beta)(\mu\alpha)}$  with the parameter replaced by  $c_{ab}$ , which cancels the third and fourth term.

Then the remaining terms in (B.16) are given by

$$\begin{aligned}
B^{(\nu\beta)(\mu\alpha)} &= \tilde{\delta}^S E^{(\nu\beta)(\mu\alpha)} - \frac{i}{8} \bar{c} \Gamma^\lambda c \partial E^{(\nu\beta)(\mu\alpha)} / \partial c^\gamma \psi_\lambda^\gamma + \frac{1}{4} (\Gamma^{bc} c)^\beta \partial \hat{\omega}_{\nu ab} / \partial \psi_\lambda^\gamma E^{(\lambda\gamma)(\mu\alpha)} \\
&\quad + \frac{1}{288} [(\Gamma^{\mu_1 \dots \mu_4})_\nu - 8 \delta_\nu^{\mu_1} \Gamma^{\mu_2 \mu_3 \mu_4}]^\beta \partial \hat{F}_{\mu_1 \dots \mu_4} / \partial \psi_\lambda^\gamma E^{(\lambda\gamma)(\mu\alpha)}. \quad (B.26)
\end{aligned}$$

This does not contain derivative operators, and therefore this cannot have terms proportional to the equation of motion. Hence

$$M^{C_{-1} B_{-1} A_{-1}}_{B_0 C_0 D_0} = 0. \quad (B.27)$$

Then from (3.29),  $B^{(\nu\beta)(\mu\alpha)}$  must be in the following form:

$$\begin{aligned} B^{(\nu\beta)(\mu\alpha)} &= R^{(\nu\beta)}_{[ab]} D^{(\mu\alpha)[ab]} + R^{(\mu\alpha)}_{[ab]} D^{(\nu\beta)[ab]} \\ &= \frac{1}{4} (\Gamma_{ab} \psi_\nu)^\beta D^{(\mu\alpha)[ab]} + \frac{1}{4} (\Gamma_{ab} \psi_\mu)^\alpha D^{(\nu\beta)[ab]}. \end{aligned} \quad (\text{B.28})$$

To confirm that (B.26) is indeed equal to (B.28), we need Fierz transformation: both expressions consist of terms containing three  $c$ 's and one  $\psi_\lambda$ . They can be rearranged to the form  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 \psi_\lambda \cdot (\Gamma_{a_1 \dots a_n} \mathcal{C})^{\alpha\beta}$  ( $n = 0, 1, \dots, 5$ ). The coefficients of  $(\Gamma_{a_1 \dots a_n} \mathcal{C})^{\alpha\beta}$  can be obtained by multiplying  $(\mathcal{C}^{-1} \Gamma^{a_1 \dots a_n})_{\beta\alpha}$  to (B.26) or (B.28). Applying (A.7) (or the faster procedure) to those coefficients we see that the difference between (B.26) and (B.28) vanishes.

## C Details of $Z$ and $W$

Let us calculate

$$Z^{A_1} = (-1)^{B_0+D_0} Z^{A_1}_{B_0 C_0 D_0 E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0}, \quad (\text{C.1})$$

$$W^{B_{-1} A_0} = (-1)^{B_0+D_0} W^{B_{-1} A_0}_{B_0 C_0 D_0 E_0} C^{E_0} C^{D_0} C^{C_0} C^{B_0}, \quad (\text{C.2})$$

by setting the symbols in the definition (4.22) and (4.23) to those given by (3.16)-(3.19), (3.21), and (B.8) (or (B.9)).

Calculation of  $Z^{A_1}$  is straightforward, except that we need Fierz identity (A.10) to cancel terms proportional to  $\psi_\mu^\alpha$ . The result is

$$Z^{A_1} = R^{A_1}_{A_2} Y^{A_2} = \partial_\mu Y, \quad (\text{C.3})$$

$$Y = -c^\mu c^\nu c^\lambda \partial_{[\mu} c_{\nu\lambda]} + \frac{i}{8} c^\mu c^\nu \bar{c} \Gamma^\lambda c A_{\mu\nu\lambda} + \frac{i}{8} c^\mu c^\nu \bar{c} \Gamma_{\mu\nu} c. \quad (\text{C.4})$$

From (3.22) and (3.32) we can easily see that some components of  $W^{B_{-1} A_0}$  vanish. Especially for  $A_0 = (A), (D)$ , and  $(S)$ ,  $B_{-1}$  must be  $(\psi)$  to give nonzero contribution. Then because  $T^{(A)}_{B_0 C_0}$  and  $T^{(D)}_{B_0 C_0}$  vanish if either  $B_0$  or  $C_0$  is  $(L)$ ,  $W^{B_{-1} A_0}$  vanishes for  $A_0 = (A)$  or  $(D)$ .

The remaining nontrivial components are  $W^{(\mu\alpha)[ab]}$  and

$$\begin{aligned} W^{[\mu\nu\lambda][ab]} &= -\frac{3}{4} i (\bar{c} \Gamma_{[\mu\nu})_\alpha D^{(\lambda)\alpha][ab]} \\ &= -\frac{3}{16} i (\bar{c} \Gamma_{[\mu\nu})_\alpha (\bar{c} \Gamma^{[a})_\beta e^{b]\rho} E^{(\lambda)\alpha)(\rho\beta)}, \end{aligned} \quad (\text{C.5})$$

$$W^{(\mu\alpha)[ab]} = \frac{i}{4} (\bar{c} \Gamma^c)_\alpha D^{(\mu\alpha)[ab]}, \quad (\text{C.6})$$

$$W^{(\mu\alpha)(\beta)} = \frac{1}{4}(\Gamma^{ab}c)^\beta D^{(\mu\alpha)[ab]} + \frac{i}{8}(\bar{c}\Gamma^\nu c)E^{(\nu\beta)(\mu\alpha)}. \quad (C.7)$$

$W^{[\mu\nu\lambda][ab]}$  is proportional to (B.13), and therefore vanishes. We can see that  $W^{(\mu\alpha)(\beta)}$  also vanishes: By using (3.21) and (B.8) or (B.9), and rearranging the resulting terms,  $W^{(\mu\alpha)(\beta)}$  is expressed by a sum of terms in the form of  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 c \cdot (\Gamma_{a_1 \dots a_n} \mathcal{C})^{\alpha\beta}$  ( $n = 0, 1, \dots, 5$ ). The coefficients of  $(\Gamma_{a_1 \dots a_n} \mathcal{C})^{\alpha\beta}$  can be obtained by multiplying  $(\mathcal{C}^{-1}\Gamma^{a_1 \dots a_n})_{\alpha\beta}$  to  $W^{(\mu\alpha)(\beta)}$ . Performing Fierz transformation (A.8) (or the faster procedure) to the coefficients we see them vanish.

It is not difficult to see that  $W^{(\mu c)[ab]}$  is in the following form:

$$W^{(\mu c)[ab]} = R^{(\mu c)}_{[de]} V^{[de][ab]} = e_\mu^d V^{[cd][ab]}, \quad (C.8)$$

where

$$\begin{aligned} V^{[cd][ab]} = & -\frac{1}{48} \frac{1}{576} e^{-1} \left[ -31 \bar{c} \Gamma^{a_1} c \bar{c} \Gamma_{a_1} c \delta_{[c}^a \delta_{d]}^b \right. \\ & + \frac{19}{2} \bar{c} \Gamma^{a_1 a_2} c \bar{c} \Gamma_{a_1 a_2} c \delta_{[c}^a \delta_{d]}^b - \frac{7}{120} \bar{c} \Gamma^{a_1 \dots a_5} c \bar{c} \Gamma_{a_1 \dots a_5} c \delta_{[c}^a \delta_{d]}^b \\ & - 6 \bar{c} \Gamma_{cd} c \bar{c} \Gamma^{ab} c + 92 \bar{c} \Gamma_{[c}^a c \bar{c} \Gamma_{d]}^b c + \frac{2}{3} \bar{c} \Gamma_{[c}^{a_1 a_2 a_3 a} c \bar{c} \Gamma_{d] a_1 a_2 a_3}^b c \\ & + 124 \bar{c} \Gamma_{[c}^a c \bar{c} \Gamma_{d]}^b c - 14 \bar{c} \Gamma^{a_1} c \bar{c} \Gamma_{a_1 cd}^{ab} c \\ & - 4 \bar{c} \Gamma^{a_1 [a} c \bar{c} \Gamma_{a_1 [c} c \delta_{d]}^b + 7 \bar{c} \Gamma^{a_1 a_2} c \bar{c} \Gamma_{a_1 a_2 cd}^{ab} c \\ & \left. - \frac{5}{3} \bar{c} \Gamma_{a_1 a_2 a_3 cd} c \bar{c} \Gamma^{a_1 a_2 a_3 ab} c + \frac{7}{6} \bar{c} \Gamma^{a_1 \dots a_4 [a} c \bar{c} \Gamma_{a_1 \dots a_4 [c} c \delta_{d]}^b \right]. \quad (C.9) \end{aligned}$$

Note that  $V^{[cd][ab]} = V^{[ab][cd]}$ . This expression is already symmetrized under interchange of four  $c^\alpha$ s. It can be shown by seeing that (C.9) is invariant if we apply (A.8). However the following reduced form given by applying (A.14), (A.15), and (A.16) to (C.9) is simpler:

$$\begin{aligned} V^{[cd][ab]} = & \frac{1}{576} e^{-1} \left[ 3 \bar{c} \Gamma^{a_1} c \bar{c} \Gamma_{a_1} c \delta_{[c}^a \delta_{d]}^b - 6 \bar{c} \Gamma_{[c}^a c \bar{c} \Gamma_{d]}^b c \right. \\ & \left. + \bar{c} \Gamma^{ab} c \bar{c} \Gamma_{cd} c - 6 \bar{c} \Gamma_{[c}^a c \bar{c} \Gamma_{d]}^b c \right]. \quad (C.10) \end{aligned}$$

Then we infer that  $W^{(\mu\alpha)[ab]}$  is given by

$$W^{(\mu\alpha)[ab]} = R^{(\mu\alpha)}_{[cd]} V^{[cd][ab]} = \frac{1}{4} V^{[ab][cd]} (\Gamma^{cd} \psi_\mu)^\alpha. \quad (C.11)$$

Indeed this is correct. From (4.23),

$$W^{(\mu\alpha)[ab]} = -\tilde{\delta} D^{(\mu\alpha)[ab]} - \frac{1}{2} \partial D^{(\mu\alpha)[ab]} / \partial c^\gamma T^{(\gamma)}$$



$$\begin{aligned}
& -\frac{1}{2}\partial T^{[ab]}/\partial c^{[cd]}D^{(\mu\alpha)[cd]} - \partial\tilde{\delta}\psi_\mu^\alpha/\partial\psi_\nu^\beta D^{(\nu\beta)[ab]} \\
& +\frac{1}{2}\partial T^{[ab]}/\partial\psi_\nu^\beta E^{(\mu\alpha)(\nu\beta)},
\end{aligned} \tag{C.12}$$

where we made a manipulation similar to  $B^{(\nu\beta)(\mu\alpha)}$  in Appendix B. Terms containing  $c^{[cd]}$  and  $c^\lambda$  cancel again by an argument similar in Appendix B. Therefore  $W^{(\mu\alpha)[ab]}$  contains terms with 4  $c^\gamma$ s and one  $\psi_\lambda$ :

$$\begin{aligned}
W^{(\mu\alpha)[ab]} &= -\tilde{\delta}^S D^{(\mu\alpha)[ab]} + \frac{i}{8}\bar{c}\Gamma^\nu c\partial D^{(\mu\alpha)[ab]}/\partial c^\gamma\psi_\nu^\gamma \\
& -\frac{1}{4}(\Gamma^{cd}c)^\alpha\partial\hat{\omega}_{\mu cd}/\partial\psi_\nu^\beta D^{(\mu\beta)[ab]} \\
& -\frac{1}{288}[(\Gamma^{\mu_1\dots\mu_4}{}_\mu - 8\delta_\mu^{\mu_1}\Gamma^{\mu_2\mu_3\mu_4})c]^\alpha\partial\hat{F}_{\mu_1\dots\mu_4}/\partial\psi_\nu^\beta D^{(\nu\beta)[ab]} \\
& +\frac{i}{8}\bar{c}\Gamma^\lambda c\partial\hat{\omega}_\lambda{}^{ab}/\partial\psi_\nu^\beta E^{(\mu\alpha)(\nu\beta)} \\
& +\frac{i}{1152}\bar{c}(\Gamma^{ab\mu_1\dots\mu_4} + 24e^{a\mu_1}e^{b\mu_2}\Gamma^{\mu_3\mu_4})c\partial\hat{F}_{\mu_1\dots\mu_4}/\partial\psi_\nu^\beta E^{(\mu\alpha)(\nu\beta)}.
\end{aligned} \tag{C.13}$$

Rearranging terms in the above into the form of  $\bar{c}\Gamma_1 c \cdot \bar{c}\Gamma_2 c \cdot \Gamma_{a_1\dots a_n}\psi_\lambda$ , and applying (A.8) (or the faster procedure) to the coefficients of  $\Gamma_{a_1\dots a_n}\psi_\lambda$ , we see that (C.11) is correct.

In summary,  $W^{B_{-1}A_0}$  is in the following form:

$$W^{B_{-1}A_0} = R^{B_{-1}}{}_{B_0}V^{B_0A_0}, \tag{C.14}$$

and the only nonzero component of  $V^{B_0A_0}$  is  $V^{(L)(L)}$  given by (C.9) or (C.10).

## D Details of $X$ and $U$

Let us calculate

$$X^{A_2} := (-1)^{C_0+E_0}X^{A_2}{}_{B_0C_0D_0E_0F_0}C^{F_0}C^{E_0}C^{D_0}C^{C_0}C^{B_0}, \tag{D.1}$$

$$U^{B_0A_0} := (-1)^{D_0+F_0}U^{B_0A_0}{}_{C_0D_0E_0F_0G_0}C^{G_0}C^{F_0}C^{E_0}C^{D_0}C^{C_0}, \tag{D.2}$$

by setting the symbols in the definition (4.34) and (4.35) to those given by (3.16)-(3.19), (3.21), (B.8) (or (B.9)), (C.4) and (C.9) (or (C.10)).

It is not difficult to see that by straightforward calculation with (A.10),

$$X^{A_2} = 0. \tag{D.3}$$

From (3.32) and (4.29), we see that some components of  $U^{B_0 A_0}$  vanish, and nontrivial components are given for  $(A_0, B_0) = (L, A), (L, D), (L, S)$  and  $(L, L)$ . Since  $T^{[\mu\nu]}$  and  $T^{(\mu)}$  do not depend on  $\psi_\mu^\alpha$  and  $c_{[ab]}$ ,  $U^{(L)(A)}$  and  $U^{(L)(D)}$  vanish. For  $(A_0, B_0) = (L, S)$ ,

$$U^{[ab](\alpha)} = \frac{1}{4} V^{[ab][cd]} (\Gamma^{cd} c)^\alpha - \frac{i}{8} \bar{c} \Gamma^\mu c D^{(\mu\alpha)[ab]}. \quad (\text{D.4})$$

By using (B.8) and (C.9) (or, (B.9) and (C.10)), we see that  $U^{[ab](\alpha)}$  consists of terms with 5  $c^\gamma$ s. They can be rearranged to terms in the form of  $(\bar{c} \Gamma_1)^\alpha \cdot \bar{c} \Gamma_2 c \cdot \bar{c} \Gamma_3 c$ . Applying (A.9) (or the faster procedure) to those terms we see that  $U^{[ab](\alpha)}$  vanishes.

For  $(A_0, B_0) = (L, L)$ ,

$$\begin{aligned} U^{[ab][cd]} &= \tilde{\delta} V^{[cd][ab]} + \frac{1}{2} \partial V^{[cd][ab]} / \partial c^\alpha T^{(\alpha)} \\ &\quad + \partial T^{[cd]} / \partial c^{ef} V^{[ef][ab]} + \partial T^{[ab]} / \partial c^{ef} V^{[ef][cd]} \\ &\quad + \partial T^{[cd]} / \partial \psi_\mu^\alpha D^{(\mu\alpha)[ab]} + \partial T^{[ab]} / \partial \psi_\mu^\alpha D^{(\mu\alpha)[cd]}, \end{aligned} \quad (\text{D.5})$$

where we made a manipulation similar to  $B^{(\nu\beta)(\mu\alpha)}$  in Appendix B. Terms containing  $c^{[ef]}$  and  $c^\mu$  cancel again by an argument similar in Appendix B. Then,

$$\begin{aligned} U^{[ab][cd]} &= \tilde{\delta}^S V^{[cd][ab]} - \frac{i}{8} \bar{c} \Gamma^\mu c \partial V^{[cd][ab]} / \partial c^\alpha \psi_\mu^\alpha \\ &\quad + \frac{i}{8} \bar{c} \Gamma^\nu c \partial \hat{\omega}_\nu^{cd} / \partial \psi_\mu^\alpha D^{(\mu\alpha)[ab]} + \frac{i}{8} \bar{c} \Gamma^\nu c \partial \hat{\omega}_\nu^{ab} / \partial \psi_\mu^\alpha D^{(\mu\alpha)[cd]} \\ &\quad + \frac{i}{1152} \bar{c} (\Gamma^{cd\mu_1 \dots \mu_4} + 24 e^{c\mu_1} e^{d\mu_2} \Gamma^{\mu_3 \mu_4}) c \partial \hat{F}_{\mu_1 \dots \mu_4} / \partial \psi_\mu^\alpha D^{(\mu\alpha)[ab]} \\ &\quad + \frac{i}{1152} \bar{c} (\Gamma^{ab\mu_1 \dots \mu_4} + 24 e^{a\mu_1} e^{b\mu_2} \Gamma^{\mu_3 \mu_4}) c \partial \hat{F}_{\mu_1 \dots \mu_4} / \partial \psi_\mu^\alpha D^{(\mu\alpha)[cd]}. \end{aligned} \quad (\text{D.6})$$

We see that  $U^{[ab][cd]}$  consists of terms in the form of  $\bar{c} \Gamma_1 c \cdot \bar{c} \Gamma_2 c \cdot \bar{c} \Gamma_3 \psi_\mu$ . By applying (A.9) (or the faster procedure), we see them cancel. In summary all the components of  $U^{B_0 A_0}$  vanish.

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